First-Year Mathematics

for

Secondary Schools

FIRST-YEAR MATHEMATICS FOR SECONDARY SCHOOLS

First-Year Mathematics
For Secondary Schools

By

GEORGE WILLIAM MYERS

Professor of the Teaching of Mathematics and Astronomy, College of Education of the University of Chicago

and

WILLIAM R. WICKES HARRIS F. MacNEISH
ERNST R. BRESLICH ERNEST A. WREIDT

Instructors in Mathematics in the University High School of the University of Chicago

School of Education Manuals
Secondary Texts

CHICAGO
THE UNIVERSITY OF CHICAGO PRESS
1907

D

First edition privately printed October, 1906
Second impression published April, 1907
Second edition August, 1907

Composed and Printed By
The University of Chicago Press
Chicago, Illinois, U. S. A.

TABLE OF CONTENTS

CHAPTER VI

USES OF THE EQUATION

CHAPTER VII

USES OF INEQUALITIES

CHAPTER VIII

THE OPERATIONS OF ARITHMETIC ABBREVIATION

CHAPTER IX

THE EVALUATION OF EXPRESSIONS

CHAPTER X

GEOMETRIC REPRESENTATION OF QUANTITY

CHAPTER XI

EQUATION APPLIED TO SIMPLE PROBLEMS ON BEAMS

CHAPTER XVI

FRACTIONS

CHAPTER XVII

EQUATIONS INVOLVING FRACTIONAL FORMS

CHAPTER XVIII

FRACTIONS INVOLVING FACTORABLE FORMS

CHAPTER XIX

FACTORING

CHAPTER XX

RATIO, PROPORTION, AND SIMILARITY

CHAPTER XXI

LINEAR EQUATIONS CONTAINING TWO UNKNOWN NUMBERS

CHAPTER XXII

QUADRATIC EQUATIONS

CHAPTER XXIII

LOGARITHMS

PREFACE

In an institution like the School of Education, which is founded upon the principle of pedagogic advance, apology for attempting an improvement in the teaching of secondary mathematics is unnecessary. A few words explanatory of the nature of the experiment which originated the following pages will however not be out of place here.

A very little study of the plan of mathematical instruction now generally in vogue in secondary schools reveals, among others, these serious weaknesses:

By giving a whole year to algebra alone before beginning geometry, as is the custom, young pupils are required to take up many very difficult matters in algebra before doing anything with even the easier and more vivid concepts of elementary geometry. The damage done to beginners by this procedure would exist in kind, though, perhaps, in less degree, if the order of algebra and geometry in the curriculum were reversed.

Such an order of subjects would make possible the growing of kindred matters of geometry, sometimes inductively, sometimes deductively, and more or less informally, out of arithmetical mensuration, and would postpone passing to the more abstract science of algebra to the second year. A unified body of mathematical truth carefully graduated as to intrinsic difficulties and touching in a broader and more vital way the interests and possibilities of youth is what is now needed and here attempted.

A second weakness of current procedure is that, without the systematic aid, either illustratively, or demonstratively, of the more graphic and more visual presentation of the similar, or analogous, mathematical ideas and truths of geometry, it requires that a whole year—and that, too, the most immature

year—be given to that particular mathematical subject which in its nature is the most abstract of all, and is most difficult of all to relate to the life-interests of boys and girls. This weakness might be somewhat mitigated—it would not be removed —by a reversal of the order of algebra and geometry. Correlation of the mathematical subjects with one another will largely eliminate it.

A third weakness is that the present scheme isolates, both as to matter and method of treatment, the mathematical subjects from each other, and from the things which to youth seem the realities of life, for so long a time during the early stages of the high-school career, as to make secondary mathematical study at once discouraging, unscientific, and unpsychological. Geometry being much more readily related to the obvious things of the world and of life, this weakness would be partially relieved by a reversal of the subjects in the curriculum. The only plan promising adequate relief is an organic permeation of the mathematical subjects with matters of real moment to modern youth and to modern human interests.

These are only a few of the ideas that actuated the mathematical faculty to take a step in organizing a body of mathematical matter out of the relevant materials of arithmetic, algebra, and geometry. This, of course, is only a correlation of the mathematical subjects among themselves. Not to depart so far from the present scheme as to render our plan of little interest to other teachers, it was determined to organize our material around an algebraic core. Accordingly, algebra very largely gives trend, unity, and character to the work of this first year.

The central purpose of this year's work is to lay a broad and solid foundation of mathematical concepts and elemental truths, and to build solidly upon them in various directions, completing very definitely a considerable body of algebra, and accomplishing meanwhile, on the side, a rounding out, from

a higher point of view, of elementary-school mathematics, and doing most of that time-consuming preliminary work necessary to induct beginners into the ideas, method, and spirit of geometry.

Some of the algebraic subjects are given only a first treatment, to be completed later, and many geometrical matters are given a sufficiently full treatment for secondary schools. Unessential details, artificial complexities, and logical overniceties are omitted, to make room for what is essential and comprehensible to beginners in mathematical reasoning. The thought-values of the work are stressed throughout, the necessary technique being made subsidiary and *auxiliary* to thinking. *Not rules, but reasons*, in the early stages of the work, and later, *when rules become necessary, rules with reasons*, are the guiding precepts.

Plan of work on manuscript.—In the preparation of the manuscript these four ideas are agreed upon by its authors:

1. Departure from the well-matured and long-practiced procedure of successful teachers is allowed only when it is substantiated by clear and cogent reasons, and can be seen to promise a distinct gain to the learner. Pure *a priori* argument is not sufficient.

2. Competent and well-supported opinion from every obtainable source, inside or outside of the faculty, as to the scientific and practical merit of proposed plans, is invited and will be fully considered with a view to acting upon it.

3. The body of the subject-matter worked out is to be regarded, not as a finished product, but merely as a *stage of study* of the problem of unifying mathematics in the secondary curriculum.

4. Youths, no less than children, look upon the doing of things as of most worth to them. Accordingly, the principles of mathematics are developed largely through the working of

problems and the generalizing of processes. Inductive methods are widely—not exclusively, nor mainly—used.

The plan, of which this book is a part, now under way in the University High School seeks to a very great extent to deduce the more important mathematical conclusions directly from the axioms. No attempt is made in the initial stages to disentangle and to render precise and consistent a necessary and sufficient body of axioms for algebra and geometry. Nor does it resort to the other alternative, so generally resorted to in the texts, of stating a necessary and sufficient number of axioms and calling upon the pupil to take them on faith. Students are at the outset allowed to use a very rich body of axioms. The body of fundamental notions with which the work proceeds for a time, is sufficient, but not necessary. After all is said, axioms and postulates are only *assumed truths*, and no harm is done in allowing many propositions, which seem obvious to the uninitiated, such as "All straight (or right) angles are equal," or "From a point on a line a perpendicular can be drawn," to be taken as axioms—they ought, indeed, to be so taken—until pupils get far enough into the subject both to know and to appreciate what the talk is about when their proof is undertaken. Later these same propositions will be proved. It should be added also that pupils must always be made clearly aware of what is being assumed in an argument.

From what has been said it appears that the pupil assists in the process of disengaging and condensing the elemental truths, and through this assistance he comes to a sense of their fundamental and necessary character. When a step in an argument, or a demonstration, is forgotten, instead of resorting to the broken chain of an author's arbitrary sequence, or, worse still, to a set proof, like a true initiate in the science, the pupil recurs to the axioms and undertakes to draw the forgotten argument from this source. The attitude of mind of the learner is at all times the attitude of the research student.

and not that of one being merely instructed. This not infrequently brings the learner upon an unexpected truth and gives him a little of the foretaste of discovery. Only the teacher can appreciate how wholesome an influence this spirit infuses into the class.

It may be well to state to any persons into whose hands it may chance to fall, that this little volume has been used in mimeograph form during the past year in the entering classes of the University High School and results were good enough to lead the mathematical faculty to desire to use it in preference to the standard texts. It is herewith printed to get the matter into better form for use in the classes of the University High School, and to hold what it contains in usable form while the second-year work is being organized.

It is hoped that the reader will regard the book merely as first-fruits of the University High School in its function as an experimental laboratory for secondary educational problems. It is not claimed to be mature fruit; but, rather, fruit that is maturing in the right direction. It is the concrete product of a genuine classroom experiment.

Criticism, suggestion, or co-operation from any competent source will be gladly welcomed to the end that the finished form, when it does appear—for an organization of a four-year high-school course in mathematics is contemplated—may be as largely and widely serviceable as possible.

In conclusion, the authors desire to render full acknowledgment to Deans W. B. Owen and H. H. Belfield for their very substantial aid in furthering this experiment, both as officials and as students of educational problems. Without their assistance the task would have been difficult; with it, the task became easy and pleasurable.

THE AUTHORS

CHICAGO, August, 1906

CHAPTER I

NUMBER GENERALIZED

§1. Uses of Positive and Negative Number

1. The top of the mercury column of a thermometer stands at 0° at the beginning of an hour. The next hour it rises 5° and the next 3°. What does the thermometer read?

2. If the mercury stands at 0°, and rises 8°, then falls 5°, what does the thermometer read?

3. Denoting a *rise* of 10°, or of x° in the thermometer by R 10°, or by R x°, and a fall of 10°, or of x°, by F 10°, or F x°, give the readings of the thermometer after the following changes, if the top of the column reads 0° at the start:

(1) R 8° followed by R 5°;
(2) R 12° " by F 9°;
(3) R 16° " by F 12°;
(4) F 13° " by F 7°;
(5) F 8° " by R 6°;
(6) F 17° " by R 10°;
(7) R 6° followed by F 5°;
(8) R 6° " by F 6°;
(9) F 6° " by R 7°;
(10) R 13° " by F 18°;
(11) F 5° " by R x°;
(12) R a° " by F b°.

4. If the change in the mercury column is a rise, a *positive* or *plus* (+) sign will be written before the number that denotes the *amount* of the change. If the change is a *fall*, a *negative* or *minus* (−) sign will be written before the number. If the reading at the start is 0°, give the readings after these changes:

(1) $+10$ followed by $+\ 2$;
(2) $+10$ " by $-\ 2$;
(3) $+20$ " by -18;
(4) -25 " by $-\ 8$;
(5) -16 " by $+16$;
(6) $+\ 9$ " by -10;
(7) -20 " by $+18$;
(8) -20 followed by $+20$;
(9) $+12$ " by -12;
(10) $+\ 9$ " by -12;
(11) $+\ x$ " by $+\ y$;
(12) $+\ a$ " by $-\ x$;
(13) $+\ a$ " by $-\ a$;
(14) $-\ a$ " by $-\ x$.

5. A bicyclist starts from a point and rides 18 miles due northward (+18 mi.) then 10 mi. due southward (−10 mi.); how far is he then from the starting point?

6. State how far and in what direction from the starting point a bicyclist would be after rides indicated by each of these pairs of records:

(1) +10 mi. then − 8 mi.;	(3) +100 mi. then +50 mi.;
(2) −20 mi. " +20 mi.;	(4) + a mi. " + b mi.

7. How far and in what direction from the starting point is a traveler who goes eastward (+) or westward (−) as shown by these pairs of numbers:

(1) +16 mi. then − 6 mi.;	(4) $+a$ mi. then $+c$ mi.;
(2) −18 mi. " +28 mi.;	(5) $+m$ mi. " $-n$ mi.;
(3) $-m$ mi. " + 3 mi.;	(6) $-m$ mi. " $+n$ mi.

8. A car in the middle of a moving train is drawn forward with a force of 8 tons and at the same time it is pulled backward with a force of $7\frac{1}{2}$ tons. The two forces together are equal to what single force?

9. Denoting a *forward* pulling force by F and a *backward* by B, give amount and direction of a single force equal to each of these pairs of forces:

(1) F 14 oz. with B 6 oz.;	(3) F 25 tons with B 15 tons;
(2) F 20 lb. " B 12 lb.;	(4) B 25 tons " F 40 tons.

10. Denoting forward-pulling by the positive or plus (+) sign and back-pulling forces by the negative or minus (−) sign, give the single force which is equal to each of these pairs of forces:

(1) +20 and −12;	(4) −15 and + 8;	(7) $+a$ and $+b$;
(2) +20 " −20;	(5) −12 " −12;	(8) $+a$ " $-b$;
(3) −15 " − 8;	(6) $+x$ " −12;	(9) $-a$ " $-b$.

11. A toy balloon pulls upward with a force of 9 oz. If a

weight of 6 oz. is attached, will the balloon rise or fall? With what force?

12. Call upward forces positive, or plus (+), and downward forces negative, or minus (−). State what single force will have the same effect as these pairs:

(1) $+17$ lb. and -7 lb.;	(4) -23 lb. and $+10$ lb.;
(2) $+17$ lb. " -10 lb.;	(5) $+x$ lb. " $+y$ lb.;
(3) -23 lb. " -10 lb.;	(6) $+x$ lb. " $-x$ lb.

13. Denoting motion *northward* by the positive or plus (+) sign and motion *southward* by the negative or minus (−) sign, and supposing a ship to start on the equator and sail as indicated, tell the latitude of the ship in both amount and sign for each pair of sailings:

(1) $+28°$ then $+2°$;	(5) $+x°$ then $-10°$;
(2) $+2°$ " $-18°$;	(6) $-x°$ " $-10°$;
(3) $+12°$ " $-12°$;	(7) $+x°$ " $-y°$;
(4) $+12°$ " $-24°$;	(8) $-x°$ " $-y°$.

14. A boy starts work with no money. He earns 50¢ (+50¢) and spends 40¢ (−40¢). How much money has he then?

15. If a man's debts be indicated by writing D before their amount and his possessions (assets) by P before their amount, what is the condition of a man's affairs if his debts and possessions are indicated by P $1,200 and D $1,000? by P $75 and D $50? D $75 and P $60? D $300 and P $,1000?

16. If water pushes (buoys) a floating body upward with a force of 18 lb., and the body's weight pulls it downward with a force of 10 lb., the two forces together equal what single force?

17. If a man was born 40 B. C. and died 45 A. D., how old was he when he died?

18. Denoting a date A. D. by + and B. C. by −, give the length of time between these pairs of dates:

(1) − 5 to +10; (3) − 52 to −50; (5) − 150 to + 150;
(2) +16 to +86; (4) −100 to +50; (6) +1600 to +1900.

19. Virgil was born −70 and died −19; how old was he at death?

20. The first Punic War lasted from −264 to −241; how long did it last?

21. Egypt was a Roman Province from −30 to +616; how many years was this?

22. Augustus was Emperor of Rome from −27 to +14; how many years was he Emperor?

23. What will denote the distance and direction from your school house to your home, if the distance and direction from your home to your school house are denoted by +60 rd.? $+1\frac{1}{2}$ mi.? $+x$ rd.? −80 rd.? $-1\frac{1}{4}$ mi.? $-a$ mi.?

24. While a freight train is moving at the rate of 10 mi. an hour toward the south (+10 mi. an hr.) a brakeman walks along the top of the cars toward the north at the rate of 4 mi. an hour (−4 mi. an hr.). How fast and in what direction does the brakeman move over the ground? Answer with the aid of the plus, or minus, sign.

25. The conductor of a passenger train walks from the front toward the rear of the train at the rate of 3 mi. an hour while the train is running at the rate of 12 mi. an hour. How fast does the conductor move over the ground? Answer with the aid of the (+), or (−), sign, supposing that + means toward the north and first, that the train is running north; then second, that the train is running south.

26. What would the sign (−) denote if the sign (+) denotes: (1) above? (2) forward? (3) upward? (4) to the right? (5) after? (6) east? (7) north? (8) possessions?

From these problems the need for distinguishing numbers

of opposite nature is evident. It is clear that the positive and negative signs afford a convenient means of making this distinction. We have seen also that letters as well as figures may be used to denote numbers.

A number having a positive or plus (+) sign before it is called a *positive* number.

What is a negative number?

The positive sign (+) need not always be written. It is generally omitted from the first number in an expression. The negative sign is never omitted. An expression like $+x-a$, where the first number, x, is positive, would commonly be written $x-a$, and is read "x minus a." The positive sign is said to be "understood" in this case.

It is now necessary to learn how to add, subtract, multiply, and divide both positive and negative numbers.

CHAPTER II

THE OPERATIONS APPLIED TO POSITIVE NUMBERS

§2. Indicating Arithmetical Operations Algebraically

1. A boy rides on his bicycle 8 mi. in one hour and 5 mi. the next hour; how far does he ride in the two hours?

NOTE.—Write the sum of 8 and 5, not 13, but $8+5$. The form $8+5$ is just as truly a sum as is 13. It will sometimes be desirable to refer to the form $8+5$ as the *indicated sum.*

2. If the boy rides a mi. the first, and 7 mi. the second hour, how far does he ride in the two hours?

3. A boy has m marbles and buys p more; how many has he then?

4. A boy had 18 marbles and lost 7; how many had he then?

NOTE.—Give the *indicated difference.*

5. A boy had m marbles and lost 8 of them; how many had he left?

6. A boy had m marbles and lost n of them; how many had he left?

7. Show the sums of these pairs of numbers and the differences, the first of the given numbers being the *minuend*, and the second, the *subtrahend:*

(1) x and 7;	(5) s and t;	(9) x and a;
(2) a " 12;	(6) 15 " n;	(10) d " c;
(3) y " 10;	(7) r " s;	(11) c " b;
(4) x " y;	(8) a " x;	(12) t " m.

8. How many yards of cloth are 12 yd. and 10 yd.?

9. How many dozens of eggs are 8 doz. and 4 doz.?

10. How many 12's are 5 12's and 4 12's?

11. How many times 12 are 9×12 and 4×12?

12. How many half-dozens are 8 half-dozens and 6 half-dozens?

13. How many times 6 are 8×6 and 6×6?

14. How many times 3 are 5×3 and 7×3 and 10×3?

15. How many times x are 2 times x and 3 times x? x and x? x and 3 times x?

NOTE.—It is customary in algebra to write $x+x$, $x+x+x$, $x+x+x+x$, etc., thus $2x$, $3x$, $4x$, and to read them "two x," "three x," "four x," and so forth.

But x times x, x times x times x, and x times x times x times x, or using the dot, which stands for the multiplication sign, $x \cdot x$, $x \cdot x \cdot x$ and $x \cdot x \cdot x \cdot x$, etc., are written x^2, x^3, x^4, etc., and are read, "x square," "x cube," "x fourth power," etc. x square may be read "x 2nd power" or "x 2nd;" x cube may be read "x 3rd power," or "x 3rd;" and, "x fourth power" may be called "x 4th," etc.

16. What would be the written form of "x 5th?" "x 6th?" "x 7th?" "x 10th?" "x nth?" What would be the meaning of each of these forms?

CAUTION.—Notice that $4x$ means $4 \cdot x$, or that x is to be used *as an addend* 4 times, while x^4 means $x \cdot x \cdot x \cdot x$, or that x is to be *used as a factor* 4 times, and similarly, for the other forms, as $3x$ and x^3, and $5x$ and x^5, etc.

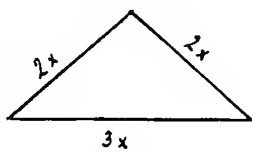

FIG. 1

17. Describe briefly the meaning of *factor* and of *addend* as they are used in arithmetic.

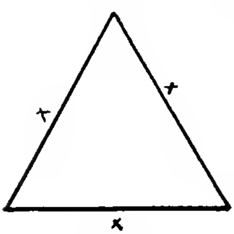

FIG. 2

18. What is the sum of the sides of a triangle whose sides are $2x$ ft., $2x$ ft., and $3x$ ft. long? Express the sum as a certain number of times x.

19. What is the sum of the three sides $2a$, $5a$, and $6a$ of a triangle?

20. A lot has the form of an equal-sided (equilateral) triangle, each side being x rd.

long. How many rods of fence will be needed to enclose it?

DEF.—Any figure whose sides are all equal is called an *equilateral* figure, as equilateral triangle, equilateral pentagon, etc.

The sum of all the sides of any closed figure is called the *perimeter* of the figure.

21. In Figure 3 what part of the *perimeter* does $x+y$ equal? What is the whole perimeter?

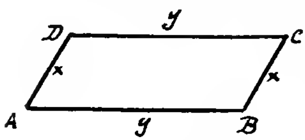

FIG. 3

22. What is the perimeter of a square of which the area is 16 sq. ft.? a^2 sq. ft.? $4a^2$ sq. ft.? x^2 sq. ft.?

23. What is the area of a square whose perimeter is 20 ft.? $4x$ ft.? $8a$ ft.?

24. Draw figures to illustrate your solutions of 23.

25. The dimensions of a rectangle are a and b, what is the perimeter? The half-perimeter? The sum of a pair of opposite sides? The sum of the other pair?

26. What is the area of the rectangle of problem 25?

Numbers denoted by letters are *literal numbers.* The product of two different literal numbers as x and y, is shown by writing the letters, or factors, side by side, as xy, with no sign between. We are familiar with the form $x \times y$ from arithmetic. The form xy is most used in algebra.

We have already seen, p. 7, that the product of x by x, or of a by a by a, etc., is written x^2, or a^3, etc.

27. What is the area of the rectangle that is 8 in. long and 5 in. wide? Of the same length and 4 in. wide? $3\frac{1}{2}$ in. wide? $6\frac{1}{4}$ in. wide?

28. What is the area of a rectangle 12 in. long and of the following widths: 6 in.? $8\frac{1}{4}$ in.? $9\frac{1}{3}$ in.? $10\frac{3}{4}$ in.? x in.? y in.?

29. What are the areas of rectangles l in. long and of the following widths: 12 in.? 9 in.? h in.? n in.? x in.? a in.?

30. Give the areas of rectangles of width w and of the following lengths: 8; 10; $12\frac{1}{2}$; x; a; l; b; z.

31. Write the products of the following pairs of factors:

(1) a by x;	(5) b by b^3;	(9) x^3 by x;
(2) b by c;	(6) a^2 by a;	(10) a by x^2;
(3) b by b;	(7) a^2 by a^3;	(11) a^2 by x;
(4) a by a^2;	(8) x by x^3;	(12) g by t^2.

32. Indicate the area of a rectangle of dimensions $a+b$ and $x+y$.

NOTE.—The product of $m+n$ and $c+d$ is written $(m+n)(c+d)$.

33. Express in terms of its base and altitude the area of the rectangle (1) of Fig. 4; of (2); of (3); of (4).

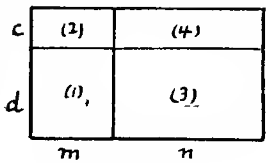

FIG. 4

34. How then may you express $(m+n)$ $(c+d)$ by using the truth that any whole equals the sum of all its parts?

35. State in words the value of $(x+y)$ $(a+b)$; $(x+y)$ $(m+n)$; $(a+x)$ $(b+y)$; $(r+s)$ $(a+x)$; $(2a+3b)(x+y)$; $(a+4x)(3b+y)$.

36. Show by a figure the value of $(a+b)(a+b)$, or $(a+b)^2$; of $(c+d)^2$; of $(x+y)^2$; of $(m+n)^2$; of $(2a+d)^2$; of $(4x+2y)^2$.

37. Each of the following expressions is the product of what two equal numbers?

(1) $a^2+2ax+x^2$;	(3) $k^2+2kb+b^2$;	(5) x^2+6x+9;
(2) $b^2+2bc+c^2$;	(4) $s^2+2st+t^2$;	(6) $c^2+8c+16$.

38. The base of a rectangle is 8 yd. and the area is 40 sq. yd. What is the altitude?

39. What is the altitude of a rectangle having—

a base=8 in. and an area=32 sq. in.?
" =8 in. " " =16 sq. in.?
" =8 in. " " =12 sq. in.?
" =5 ft. " " =$7\frac{1}{2}$ sq. ft.?

40. What is the base of a rectangle having—

an altitude	$=9$ ft.	and	an area	$=27$ sq. ft.?
"	$=9$ ft.	"	"	$=18$ sq. ft.?
"	$=9$ ft.	"	"	$=15$ sq. ft.?
"	$=9$ ft.	"	"	$=12$ sq. ft.?
"	$=9$ ft.	"	"	$=6$ sq. ft.?

41. What is the other dimension of a triangle having—

a base	$=6$ ft.	and	an area	$=24$ sq. ft.?
"	$=6$ ft.	"	"	$=12$ sq. ft.?
"	$=6$ ft.	"	"	$=9$ sq. ft.?
"	$=6$ ft.	"	"	$=3$ sq. ft.?
an altitude	$=4$ yd.	and	an area	$=16$ sq. yd.?
"	$=4$ yd.	"	"	$=8$ sq. yd.?
"	$=4$ yd.	"	"	$=4$ sq. yd.?
"	$=a$ ft.	"	"	$=a$ sq. ft.?
"	$=h$ rd.	"	"	$=hl$ sq. rd.?
"	$=b$ in.	"	"	$=ab$ sq. in.?
"	$=b$ in.	"	"	$=by$ sq. in.?
"	$=c$ in.	"	"	$=a$ sq. in.?
"	$=a$ in.	"	"	$=b$ sq. in.?

The quotient of x divided by y is written $\frac{x}{y}$, or $x \div y$, and read "x divided by y." $\frac{x}{y}$ is also read "x over y."

42. Write the quotient of the first of these numbers divided by the second:

(1) m and n;	(6) $a+b$ and $c+d$;	(11) a^2-b^2* and $a+b$;
(2) c " n;	(7) $4x$ " $3y$;	(12) a^2-b^2 " $a-b$;
(3) a " b;	(8) $3x$ " $4y$;	(13) x^2 " $a+b$;
(4) b " a;	(9) $a+b$ " c;	(14) $(a+b)^2$ " $a+b$;
(5) x " y;	(10) a " $x+y$;	(15) $(a-b)^2$ " $a-b$.

*Note. a^2-b^2 is read "a square minus b square;" $(a-b)^2$ is read "the square of $a-b$," and $(a+b)^2$ is read "the square of $a+b$."

CHAPTER III

THE ARITHMETICAL OPERATIONS WITH NUMBERS REPRESENTED BY LINES

§ 3. Operations with Lines

Sums, differences, products, and quotients may be constructed. For notebook work in construction a ruler and a pair of compasses are needed. For blackboard work the compasses may be replaced by chalk and string.

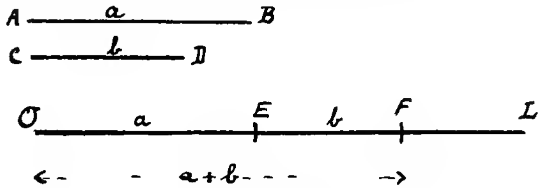

FIG. 5.

1. To construct the *sum* of two lines, for example a and b (Fig. 5).

Construction.—Draw a straight line, as OL, longer than a and b together when placed end to end. Open the compasses so that the points will be a distance apart equal to a. With the pin-point at O mark a short arc across OL at E, with the pencil point.

Then with the distance, b, between the points of the compasses and with the pin-point on E, mark a second arc, as at F, across OL. What line then has the length equal to $a+b$? Would the sum be the same if the lines a and b were added in the reverse order?

2. Draw a pair of lines in your notebook and on another line construct their sum.

REMARK.—The sum of two lines of different lengths on the blackboard may be constructed with crayon and string.

3. Draw lines to represent $2a$ and $3b$ and show how to construct $2a+3b$.

4. Construct $m+2d$; $x+y+z$; $x+2y+2z$; $2(a+b)$; $3(a+b)$; $3a+3b$.

5. Compare the sums $2a+2b$ and $2(a+b)$; $3(a+b)$ and $3a+3b$.

6. To construct the *difference* of two lines, proceed as with the sum until the subtrahend line is to be marked off. Instead of marking D on beyond E on OL allow D to fall on the other side of E, i. e., back on the minuend line. Point out in the adjoining figure (Fig. 6), the line $a-b$.

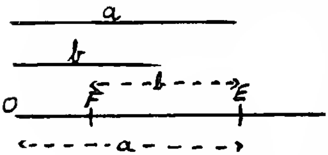

FIG. 6

7. Construct the differences of these pairs of lines, the first being the *minuend* line:

(1) m and n, $(m>n)$;*
(2) a " x, $(a>x)$;
(3) c " d, $(c>d)$;
(4) p " x, $(p>x)$;
(5) x and y, $(x=y)$;
(6) c " d, $(c=d)$;
(7) $2x$ and $2x$;
(8) y and y.

*NOTE: $a>b$ indicates that a is greater than b.

Notice that if the subtrahend line is the longer the point D will fall beyond O (i. e., to the left on the figure). Whenever the difference (OD) is measured, or extends, toward the right of O (or along the minuend line) it will be positive; when it extends toward the left from O, i. e., on the minuend line prolonged through O, it will be negative.

8. Show how to subtract 4 from 3; 8 from 5; 6 from 3; 7 from 2; 9 from 1; 3 from 0; and state both the magnitude and the sign (*direction*) of the differences.

9. Construct the following differences; the first number denoting the minuend line:

(1) $3x$ and x;
(2) $2x$ " $3x$;
(3) x " $5x$;
(4) $2a$ and $2b$;
(5) $6a$ " $4a$;
(6) $3a$ " $4a$;
(7) a and $4a$;
(8) $4m$ and $6m$.

10. Construct these expressions:

(1) $a+x-y$;
(2) $a+2x-2y$;
(3) $2(a-x+y)$;
(4) $2(a+b-2c)$;
(5) $2(a-b+2c)$;
(6) $3a-2b-a$.

§ 4. Sums and Differences of Angles

1. Two angles, as x and y, Fig. 7, may be added by placing together one side, as AC of angle y, along a side, as AC of the other angle, x. If AD lies on the side of AC opposite to AB, the angle between AB and AD will be the angle $x+y$.

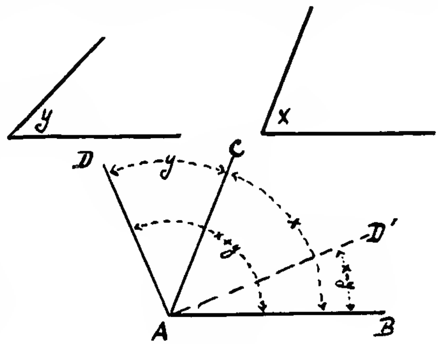

FIG. 7

2. If the angle, y, is turned over AC as a hinge, making AD to lie along the dotted line AD′, the angle between AD′ and AB is the difference, $x-y$.

3. Draw two angles, a and b, on the black board, and show how to obtain their sum without measuring the angles. Show how to obtain their difference without measuring.

4. Show the sum and the difference of two angles by folding or cutting paper.

§ 5. Products as Rectangles

The product of two numbers may be constructed as follows:

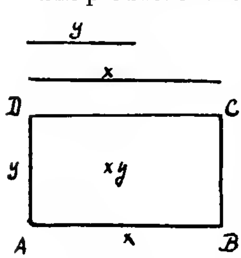

FIG. 8

Let the two numbers be represented by lines, as x and y, in Fig. 8.

Construction: At the end A of AB $(=x)$ draw a perpendicular line AD and make it equal to y in length. Through D draw DC parallel to AB and through B draw BC parallel to AD. The rectangle ABCD represents the product of x and y. The rectangle denotes the product, in the sense that it contains as

many square units of area as there are units in the number xy.

1. Show the product of two lines of given lengths on the blackboard.

2. Draw the product of one or more pairs of lines in your notebook.

3. Construct the product of two lines $2x$ and x, of $3x$ and $2x$.

CHAPTER IV

ADDITION, SUBTRACTION, AND MULTIPLICATION OF POSITIVE AND NEGATIVE WHOLE NUMBERS

§ 6. Adding Positive and Negative Whole Numbers

1. Denoting distances traveled northward by positive numbers, and distances traveled southward by negative numbers, find for each of the following cases the distances and the direction of the stopping point from the starting point. When the stopping point is north of the starting point mark the result $+$; when south, mark the result $-$.

An automobile goes:

(1)	$+15$ mi., then -10 mi.;	(6)	-12 mi., then -10 mi.;
(2)	$+15$ mi., " -14 mi.;	(7)	-20 mi., " $+15$ mi.;
(3)	$+15$ mi., " -20 mi.;	(8)	-15 mi., " $+22$ mi.;
(4)	$+25$ mi., " -35 mi.;	(8)	-20 mi., " $+21$ mi.;
(5)	-18 mi., " $+24$ mi.;	(10)	-22 mi., " $+22$ mi.

2. In the following problems the numbers indicate distances traveled northward, if negative; and southward, if positive. The *sum in all cases must denote the distance and direction* of the stopping point from the starting point. Write the sums with their proper signs:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
$+15$	-15	$+15$	-15	$+38$	-38	$+38$	-38
$+\ 8$	$-\ 8$	$-\ 8$	$+\ 8$	$+19$	-19	$+19$	$+19$

(9)	(10)	(11)	(12)	(13)
$+\ 4$	$-\ 4$	$+11$	$-\ 4$	$+12$
$+26$	-26	-26	$+26$	-12.

3. Examine (1), (2), (5), (6), (10) of problem 2 and make a rule for adding two numbers having like signs.

4. From (3), (4), (8), (11), (12), and (13) of problem 2 make a rule for adding two numbers having *unlike* signs.

Sums, with their proper signs, of positive and negative numbers, are called algebraic sums. The sums and differences of numbers regardless of sign, are called arithmetical sums and differences.

SUMMARY

The algebraic sum of two numbers with like signs is their arithmetical sum, with the common sign prefixed.

The algebraic sum of two numbers with unlike signs is their arithmetical difference, with the sign of the larger number prefixed.

5. In the following problems the positive numbers indicate gains and the negative numbers indicate losses. The sums indicate the net change in the man's capital, and whether the net change is an increase or a decrease. Find the sums and tell their meaning:

(1)	(2)	(3)	(4)	(5)	(6)	(7)
$+50$	$+35$	-45	$+75$	-236	$+8x$	$-14a$
$+25$	-38	-20	$+13$	$+780$	$-6x$	$-46a$
-18	$+24$	$+60$	-86	$-\ 95$	$-4x$	$+77a$
$-\ 6$	-15	$+55$	$+\ 8$	$+\ 45$	$+7x$	$-\ 5a$

6. State a way of adding any number of positive and negative numbers.

7. A force of 12 lb. pulling toward the right (+12 lb.) together with a force of 9 lb. pulling toward the left give a combined pull equal to what force?

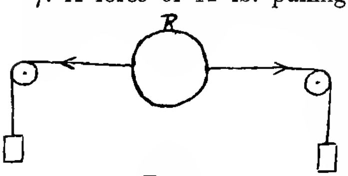

FIG. 9

8. What single force has the same effect in pulling the ring R as the following pairs of forces acting together?

(1) $+12$ lb. and $-\ 8$ lb.;	(9) -16 lb. and $-\ 8$ lb.;
(2) -12 lb. " $+\ 8$ lb.;	(10) $+\ 3$ lb. " -12 lb.;
(3) -10 lb. " $+10$ lb.;	(11) $+\ x$ lb. " $+\ y$ lb.;
(4) -16 lb. " $+13$ lb.;	(12) $+\ x$ lb. " $-\ y$ lb.;
(5) $+14$ lb. " -17 lb.;	(13) $-\ x$ lb. " $+\ y$ lb.;
(6) $+\ 9$ lb. " -20 lb.;	(14) $-\ x$ lb. " $-\ y$ lb.;
(7) $+11$ lb. " $+15$ lb.;	(15) $+\ x$ lb. " $-\ x$ lb.;
(8) -16 lb. " $+13$ lb.;	(16) $-2x$ lb. " $+\ x$ lb.

9. A man draws a bucket of brick, weighing 60 lb., to a house top by pulling on a rope which runs over a pulley, with a force of 65 lb. What single force equals the sum of the two forces acting on the handle of the bucket?

10. A balloon pulls upward on a stone, weighing 6 oz., with a force of 8 oz. What is the sum of the forces?

11. A piece of iron weighing 18 lb., when placed under water, is pushed (buoyed) upward with a force of $2\frac{4}{7}$ pounds. What is the sum (combined effect) of the two forces together?

12. An elevator starts at a certain floor, goes up 65 ft., down 91 ft., up 52 ft., down 13 ft., and up 65 ft., and stops. How far and in what direction is the stopping from the starting point? Give your answer in the form of an algebraic sum.

13. A vessel starts in latitude $+20°$, it sails $+13°$ in latitude, then $-60°$, then $+40°$, then $-10°$. What is its latitude after the sailings? What is the latitude of a ship starting in latitude $-50°$ after these changes of latitude: $+10°$, $-5°$, $+18°$, $-7°$, $+38°$, $-12°$, $+60°$?

14. A boatman rows at a rate that would carry him 3 miles an hour through still water, down a river whose current is 2 mi. an hour. What is his rate per hour? What would be his rate per hour, if he rows up the river?

§ 7. Subtracting Positive and Negative Whole Numbers

1. In this problem positive numbers indicate the readings above zero and negative numbers, readings below zero. The

difference means the number of degrees the top of the mercury column must rise, or fall, to change from the second reading to the first. If the change is a *rise*, mark it positive (+), if a fall, mark it negative (−).

	(1)	(2)	(3)	(4)	(5)
First reading:	$+68°$	$-98°$	$-30°$	$-7°$	$+1°$
Second reading:	$+42°$	$-18°$	$+65°$	$+32°$	$-28°$

	(6)	(7)	(8)	(9)	(10)
First reading:	$0°$	$-8°$	$+6x°$	$-4a°$	$-2y°$
Second reading:	$+78°$	$+8°$	$-3x°$	$-9a°$	$+60y°$

Define minuend. Define subtrahend.

2. By the summary, p. 16, find the sums in the following problems and compare the exercises and your results with those of the like numbered exercises of problem 1:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
$+68$	-98	-30	-7	$+1$	0	-8	$+6x$	$-4a$	$-2y$
-42	$+18$	-65	-32	$+28$	-78	-8	$+3x$	$+9a$	$-60y$

3. Show by comparing the problems of 1 and 2 that the difference of any two numbers can be found by changing the sign of the subtrahend and then adding.

4. Find the differences of the following:

(1)	(2)	(3)	(4)	(5)	(6)	(7)
$+19$	-60	-75	$+8$	$-3a$	$+18x$	$-12y$
-10	-25	$+25$	-16	$-2a$	$-6x$	$-7y$

(8)	(9)	(10)	(11)
$+a$	$+3x-18$	$+a+2b-c$	$-12a+3x$
$-b$	$+2x+6$	$+3a-5b+3c$	$+7a-2x$

§ 8. Multiplication of Positive and Negative Whole Numbers

Suppose the short spaces on the line east-west, E W, represent a mile.

1. Show what space starting from 0 in each case represents $+2$ mi.; $+3$ mi.; $+5$ mi.; $+8$ mi.; $+10$ mi.

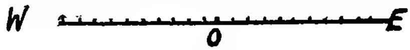

FIG. 10

2. Starting again from 0, show the space that represents -1 mi.; -2 mi.; -5 mi.; -7 mi.; -10 mi.

3. Show the spaces from 0 a man goes if he travels $+2$ mi. a day for 1 day; 2 days; 3 days; 5 days.

4. What is 2 times -2 mi.? 3 times -2 mi.? 4 times -2 mi.? 5 times -2 mi.?

5. What is 5 times -4 mi.? -10 mi.? -20 mi.? -25 mi.? -100 mi.?

6. What is 16 times -2 mi.? -5 mi.? -10 mi.? -20 mi.? -100 mi.?

7. What is $50\times(-2)$? $25\times(-10)$? $40\times(-12)$? $8\times(-120)$? $4\times(-a)$? $10\times(-x)$?

8. The value of a man's property changes by $+1000$ a year. How much does it change in 2 yr.? 5 yr.? 8 yr.? 10 yr.? In each case tell whether the change is an increase, or a decrease.

9. If the value of a man's property changes by $-\$500$ a year, how much does it change in 3 yr.? 5 yr.? 7 yr.? 8 yr.? 10 yr.? 12 yr.? In each case tell whether the change is an increase, or a decrease.

10. How much and in what way does a man's property change in 12 yr. at the rate of \$50 a yr.? $-\$100$ a yr.? $+\$400$ a yr.? $+\$1,800$ a yr.? $-\$2,000$ a year?

11. How much and in what direction does the height of a mercury column of a thermometer change in 6 hours at the rate of $+10°$ an hr.? $-8°$ an hr.? $-7°$ an hr.? $+5\frac{1}{2}°$ an hr.? $-3\frac{1}{2}°$ an hr.? $+2\frac{1}{2}°$ an hr.? $-a°$ an hour?

12. State a way of multiplying a negative number by an arithmetical number.

The bar, AB, is balanced on the middle peg, marked O. There are 5 equally spaced pegs on either side of O. Equal weights provided with hooks may be hung on the pegs. A smooth rod, CD, fitted with a pulley, P, over which a cord passes, is held in such position that weights may be hung at one end of the cord. The other end of the cord may be hooked to

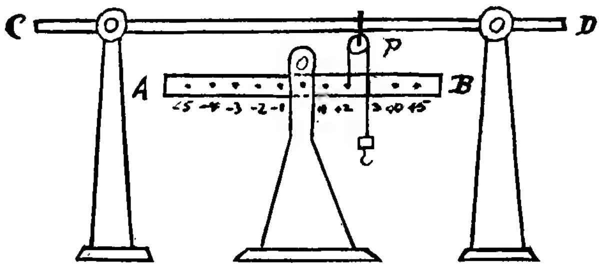

FIG. 11

any of the pegs. By moving the sliding pulley along, the force of the weight may be made to pull vertically upon the peg. Notice that with this apparatus we may use forces acting either upward or downward, on pegs either to the right or left of the center. The small weights are all equal and the spaces between the pegs are all equal.

Call a weight w oz., and a space x inches.

13. How shall we distinguish distances measured to the right of O from those measured to the left?

14. How shall we distinguish forces pulling upward from those pulling downward?

15. If the bar is balanced and a weight is then hung at $+1$, what will occur? What will occur if the weight, w, is hung on peg $+2$? $+3$? $+4$? $+5$? -2? -3? -4? -5?

16. Suppose that no weights are hung to the pegs, but that

a weight hangs from the end of the cord and pulls upward on peg $+1$; what will occur? What will occur if the weight pulls upward on peg $+3$? $+5$? -1? -2? -3? -5?

17. Downward forces on any peg to the right of O have a tendency to turn the bar around O in what direction? What turning tendency do downward forces have on any peg to the left? Upward forces on positive pegs (pegs to the right)? Upward forces on negative pegs?

18. What two turning tendencies must we deal with?

NOTE.—Imagine a watch laid down on the drawing, face up, with the hand post just over O. When the bar turns, or tends to turn, around *with* the hands of the watch, the bar is said to have a negative, or minus $(-)$, turning tendency. If the tendency is *against* (opposite to) that of the watch hands, it is a positive, or plus $(+)$, tendency.

DEFINITION.—The distance from the turning point, O, to the point where the force or weight acts is called the *lever arm*, or *arm* of the force.

The turning tendency of any weight or force is the product of the number of units in the force by the number of units in the *arm* of the force. Or $T = f \times a$, where T, is the turning tendency, f the force, and a the arm.

19. What is the turning tendency of a weight, or force, $-2w$ ($2w$ pulling downward) on the peg $+12$? Of a force $+w$ on the peg $+2$?

SOLUTION.—(1) The turning tendency, T, is $(-2w)\times(+1x)$; since the force is $-2w$ and the arm is $+1x$, x, or $+x$. $2w\times x = 2wx$, and as the bar tends to turn *with* the hands of the watch, $T = -2wx$. (2) $T = (+w)\times(+2x) = +2wx$ ($+$ for what reason?).

20. What is the turning tendency of $+w$ at $+3$? $+2w$ at $+2$? $+3w$ at $+4$? $-w$ at $+3$? $-2w$ at $+4$? $-3w$ at -1? $-2w$ at -3? $-2w$ at -4? $+2w$ at -2? $+3w$ at -2? $+2w$ at -4?

21. What sign (direction) has the turning tendency of a plus force with a plus arm? A plus force with a minus arm? A minus force with a plus arm? A minus force with a minus arm?

22. If 3 (-4) means that -4 is to be measured, or laid off, 3 times from o in the negative direction, what is the value

FIG. 12

of the product, $3(-4)$, or, what is the same thing, $(+3)(-4)$? The product means the distance and direction this leaves us from o, evidently -12.

23. Show on a figure the meaning and value of the following products:

(1) $2(+3)$;	(3) $3(-2)$;	(5) $3(-5)$;	(7) $2(+6)$;
(2) $2(-3)$;	(4) $3(+2)$;	(6) $3(+5)$;	(8) $2(-6)$.

24. Show on a figure the product $(-3)(-4)$.

NOTE. $(-3)(-4)$ means that four is to be laid off 3 times in the direction opposite to the direction of -4, i. e., *in the positive direction.*

25. Interpret these products on the same principle.

(1) $(-2)(-3)$;	(4) $(-2)(-8)$;	(7) $(-2)(+5)$;
(2) $(+3)(-2)$;	(5) $(-3)(-5)$;	(8) $(-2)(-5)$;
(3) $(-2)(+4)$;	(6) $(+3)(-5)$;	(9) $(-3)(+6)$.

26. Show on Fig. 13 the value of 2 times $+a$, $2(-a)$; $(-2)(+a)$; $(-2)(-a)$; $(-3)(-a)$; $(-3)(+a)$; $(-4)(-a)$.

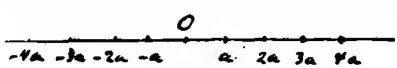

FIG. 13

27. Interpret the meanings of:

(1) $(+3)(+a)$;	(5) $(+a)(+b)$;	(9) $(+c)(-d)$;
(2) $(+3)(-a)$;	(6) $(+a)(-b)$;	(10) $(+c)(+d)$;
(3) $(-3)(+x)$;	(7) $(-a)(+b)$;	(11) $(-c)(+d)$;
(4) $(-3)(-x)$;	(8) $(-a)(-b)$;	(12) $(-c)(-d)$.

SUMMARY

In digits:	*In letters:*
$(+3)(+4)=+12$;	$(+a)(+b)=+ab$;
$(+3)(-4)=-12$;	$(+a)(-b)=-ab$;
$(-3)(+4)=-12$;	$(-a)(+b)=-ab$;
$(-3)(-4)=+12$;	$(-a)(-b)=+ab$.

28. Examine the eight products of the summary and make a rule for obtaining the algebraic sign of a product of two numbers from the signs of the factors.

Compare your rule with this:

§ 9. Law of Signs for Multiplication

If two factors have the same sign, their product is positive; and if two factors have unlike signs, their product is negative.

CHAPTER V

OPERATIONS ON FRACTIONAL NUMBERS GENERALIZED

The operations of addition, subtraction, multiplication, and division must frequently be used upon fractions.

§ 10. Unit-Fractions—Review-Problems

1. What is the sum of $\frac{1}{2}$ and $\frac{1}{3}$? $\frac{1}{5}$ and $\frac{1}{7}$? $\frac{1}{6}$ and $\frac{1}{9}$? $\frac{1}{10}$ and $\frac{1}{15}$?

2. What is the sum of $\frac{1}{2}$ and $\frac{1}{7}$? $\frac{1}{5}$ and $\frac{1}{8}$? $\frac{1}{12}$ and $\frac{1}{15}$?

Explanation of $\frac{1}{2}+\frac{1}{7}$: $\frac{1}{2}+\frac{1}{7}=\frac{7}{2\cdot 7}+\frac{2}{2\cdot 7}=\frac{7+2}{7\cdot 2}$.

In the same way write out the work of all the problems 1 and 2.

3. Examine your results, compare them with the numbers in the given problem and make a rule for quickly adding fractions whose numerators are 1.

Definition.—Fractions having 1 for numerators are called *unit* fractions, or fractional units.

4. Apply your rule to these sums:

(1) $\frac{1}{3}+\frac{1}{4}$;
(2) $\frac{1}{5}+\frac{1}{6}$;
(3) $\frac{1}{2}+\frac{1}{5}$;
(4) $\frac{1}{7}+\frac{1}{9}$;
(5) $\frac{1}{3}+\frac{1}{x}$;
(6) $\frac{1}{9}+\frac{1}{b}$;
(7) $\frac{1}{x}+\frac{1}{y}$;
(8) $\frac{1}{n}+\frac{1}{m}$.

5. What is the difference of $\frac{1}{2}$ and $\frac{1}{3}$? $\frac{1}{4}$ and $\frac{1}{5}$? $\frac{1}{5}$ and $\frac{1}{7}$? $\frac{1}{6}$ and $\frac{1}{9}$? $\frac{1}{10}$ and $\frac{1}{12}$? $\frac{1}{2}$ and $\frac{1}{15}$?

Explanation of $\frac{1}{5}-\frac{1}{7}$: $\frac{1}{5}-\frac{1}{7}=\frac{7}{5\cdot 7}-\frac{5}{5\cdot 7}=\frac{(7-5)}{5\cdot 7}$.

Note.—Write out the work of all the parts of problems 5 as is done in this explanation.

6. Examine your results and make a rule for quickly finding the difference of unit fractions.

7. Apply your rule to finding these differences:

(1) $\frac{1}{3}-\frac{1}{5}$; (2) $\frac{1}{4}-\frac{1}{7}$; (3) $\frac{1}{9}-\frac{1}{x}$; (4) $\frac{1}{3}-\frac{1}{y}$; (5) $\frac{1}{b}-\frac{1}{x}$; (6) $\frac{1}{c}-\frac{1}{b}$; (7) $\frac{1}{s}-\frac{1}{p}$; (8) $\frac{1}{t}-\frac{1}{m}$; (9) $\frac{1}{a}-\frac{1}{b}$.

8. Show that the addition and the subtraction of unit fractions may be indicated thus:

$$\frac{1}{5}\pm\frac{1}{11}=\frac{11\pm5}{5\cdot11}.$$

9. Solve the following problems:

(1) $\frac{1}{4}\pm\frac{1}{7}$; (2) $\frac{1}{7}\pm\frac{1}{10}$; (3) $\frac{1}{6}\pm\frac{1}{10}$; (4) $\frac{1}{11}\pm\frac{1}{15}$; (5) $\frac{1}{x}\pm\frac{1}{y}$; (6) $\frac{1}{a}\pm\frac{1}{x}$; (7) $\frac{1}{d}\pm\frac{1}{c}$; (8) $\frac{1}{m}\pm\frac{1}{n}$; (9) $\frac{1}{s}\pm\frac{1}{t}$; (10) $\frac{1}{t}\pm\frac{1}{r}$.

§ 11. Fractions Having the Same Numerator

1. Add $\frac{3}{5}$ and $\frac{3}{7}$.

SOLUTION: $\frac{3}{5}+\frac{3}{7}=3(\frac{1}{5}+\frac{1}{7})=3\frac{(7+5)}{7\cdot5}=3\cdot\frac{12}{35}=1\frac{1}{35}$.

2. Subtract $\frac{3}{7}$ from $\frac{3}{4}$.

SOLUTION: $\frac{3}{4}-\frac{3}{7}=3(\frac{1}{4}-\frac{1}{7})=3\frac{7-4}{7\cdot4}=\frac{3\cdot3}{7\cdot4}=\frac{9}{28}$.

3. Solve the following exercises:

(1) $\frac{2}{3}\pm\frac{2}{5}$; (2) $\frac{3}{5}\pm\frac{3}{8}$; (3) $\frac{5}{6}\pm\frac{5}{9}$; (4) $\frac{6}{7}\pm\frac{6}{11}$; (5) $\frac{n}{7}\pm\frac{n}{11}$; (6) $\frac{c}{x}\pm\frac{c}{12}$; (7) $\frac{a}{b}\pm\frac{a}{c}$; (8) $\frac{x}{y}\pm\frac{x}{z}$; (9) $\frac{s}{t}\pm\frac{s}{r}$ (10) $\frac{t}{s}\pm\frac{t}{n}$.

§ 12. Addition and Subtraction of Fractions in General

1. Add $\frac{5}{6}$ and $\frac{7}{8}$.

SOLUTION: $\frac{5}{6}+\frac{7}{8}=\frac{5\cdot8}{6\cdot8}+\frac{6\cdot7}{6\cdot8}=\frac{5\cdot8+6\cdot7}{6\cdot8}$.

2. Subtract $\frac{4}{7}$ from $\frac{8}{9}$.

SOLUTION: $\frac{8}{9}-\frac{4}{7}=\frac{7\cdot 8}{7\cdot 9}-\frac{4\cdot 9}{7\cdot 9}=\frac{7\cdot 8-4\cdot 9}{7\cdot 9}$.

3. Add $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ where n_1 (read "n one"), d_1 ("d one") stand for a first numerator and a first denominator n_2 (read "n two") and d_2 (read "d two") stand for a second numerator and a second denominator, respectively.

4. Subtract $\frac{n_2}{d_2}$ from $\frac{n_1}{d_1}$. The numbers n_1, d_1, n_2, and d_2 indicate the same as in problem 3.

NOTE.—Observe that the subtraction of $\frac{n_2}{d_2}$ from $\frac{n_1}{d_1}$ is indicated thus: $\frac{n_1}{d_1}-\frac{n_2}{d_2}$.

5. Make a rule for finding the sum, or the difference, of any two fractions.

6. Write, by the rule just made, the values of these indicated sums and differences:

(1) $\frac{2}{7}+\frac{1}{11}$;
(2) $\frac{5}{8}+\frac{4}{11}$;
(3) $\frac{6}{7}-\frac{3}{4}$;
(4) $\frac{7}{12}+\frac{4}{11}$;
(5) $\frac{5}{13}\pm\frac{2}{11}$;
(6) $\frac{7}{10}\pm\frac{4}{9}$;
(7) $\frac{12}{17}\pm\frac{5}{9}$;
(8) $\frac{x}{3}\pm\frac{7}{8}$;
(9) $\frac{a}{9}\pm\frac{7}{12}$;
(10) $\frac{a}{5}\pm\frac{c}{10}$;
(11) $\frac{m}{8}\pm\frac{x}{3}$;
(12) $\frac{x}{a}\pm\frac{y}{a}$;
(13) $\frac{n_1}{d_1}\pm\frac{n_2}{d_2}$;
(14) $\frac{8}{x}\pm\frac{7}{y}$;
(15) $\frac{x}{y}\pm\frac{c}{d}$.

§ 13. Multiplication and Division of Fractions in General

1. Multiply $\frac{3}{4}$ by 4; by 8; by 12; by 40; by 5; by 10; by 25; by 200; by a.

2. Multiply $\frac{3}{4}$ by $\frac{4}{5}$; by $\frac{8}{9}$; by $\frac{3}{5}$; by $\frac{4}{3}$; by $\frac{8}{3}$; by $\frac{12}{3}$; by $\frac{x}{3}$; by $\frac{a}{3}$.

3. Solve the following:

(1) $\frac{2}{3}\times\frac{4}{5}$; (2) $\frac{1}{4}\times\frac{2}{3}$; (3) $\frac{4}{5}\times\frac{8}{9}$; (4) $\frac{5}{6}\times\frac{7}{8}$; (5) $\frac{7}{8}\times\frac{2}{5}$; (6) $\frac{a}{5}\times\frac{b}{7}$;

(7) $\frac{a}{b}\times\frac{c}{d}$; (8) $\frac{x}{y}\times\frac{c}{d}$; (9) $\frac{s}{t}\times\frac{p}{q}$;

(10) $\frac{a}{x}\times\frac{c}{d}$; (11) $\frac{n_1}{d_1}\times\frac{n_2}{d_2}$; (12) $\frac{a_1}{b_1}\times\frac{a_2}{b_2}$.

4. Make a rule for multiplying any two fractions together.

5. Apply your rule to these indicated products: (A dot, thus (·), also denotes multiplication; as $4 \cdot 4=20$.)

(1) $\frac{2}{3}\cdot\frac{3}{2}$; (2) $\frac{3}{5}\cdot\frac{5}{3}$; (3) $\frac{4}{7}\cdot\frac{7}{4}$; (4) $\frac{5}{8}\cdot\frac{8}{5}$; (5) $\frac{7}{8}\cdot\frac{8}{7}$; (6) $\frac{5}{6}\cdot\frac{6}{5}$; (7) $\frac{7}{9}\cdot\frac{9}{7}$;

(8) $\frac{x}{3}\cdot\frac{3}{x}$; (9) $\frac{x}{10}\cdot\frac{10}{x}$; (10) $\frac{x}{y}\cdot\frac{y}{x}$; (11) $\frac{c}{d}\cdot\frac{d}{c}$;

(12) $\frac{n_1}{d_1}\cdot\frac{d_1}{n_1}$; (13) $\frac{2}{1}\cdot\frac{1}{2}$; (14) $\frac{4}{1}\cdot\frac{1}{4}$; (15) $\frac{12}{1}\cdot\frac{1}{12}$; (16) $\frac{a}{1}\cdot\frac{1}{a}$;

6. Find the values of these products:

(1) $\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}$; (2) $\frac{3}{4}\cdot\frac{5}{7}\cdot\frac{7}{8}$; (3) $\frac{3}{4}\cdot\frac{5}{7}\cdot\frac{a}{b}$;

(4) $\frac{5}{6}\cdot\frac{x}{y}\cdot\frac{s}{t}$; (5) $\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{x}{y}$;

(6) $\frac{n_1}{d_1}\cdot\frac{n_2}{d_2}\cdot\frac{n_3}{d_3}$; (7) $\frac{a_1}{b_1}\cdot\frac{a_2}{b_2}\cdot\frac{a_3}{b_3}$.

7. Make a rule for multiplying any three fractions together.

8. Change your rule to make it apply to the product of 4 fractions; of *n* fractions.

9. What is the simplest form of the product in each case of problem 5?

10. By examining the products of problem 5, tell without actually dividing, how many times $\frac{2}{3}$ is contained in 1; $\frac{3}{2}$ in 1; $\frac{3}{5}$ in 1; $\frac{5}{3}$ in 1; $\frac{4}{7}$ in 1; $\frac{7}{4}$ in 1; $\frac{x}{10}$ in 1; 2 in 1; 12 in 1;

$\frac{5}{3}$ in 1; a in 1; $\frac{7}{9}$ in 1; $\frac{9}{7}$ in 1; $\frac{a}{b}$ in 1; $\frac{b}{a}$ in 1.

11. Give a quick way of finding how many times any fraction, or any whole number, is contained in 1.

PRINCIPLE I.—*The reciprocal of a number shows how many times it is contained in* 1.

12. How many times are these numbers contained in 1?

(1) $\frac{4}{9}$? (2) $\frac{6}{7}$? (3) $\frac{7}{8}$? (4) $\frac{6}{11}$? (5) $\frac{x}{3}$? (6) $\frac{3}{x}$? (7) $\frac{a}{b}$? (8) $\frac{x}{y}$? (9) $\frac{a}{x}$?

13. A certain number is contained in 1, $\frac{3}{5}$ times; how many times is it contained in 2? in 3? in 5? in 8? in 20? in a? in c? in $\frac{3}{4}$? in $\frac{5}{9}$? in $\frac{7}{8}$? in $\frac{3}{16}$? in $\frac{a}{b}$? in $\frac{c}{d}$? in $\frac{x}{y}$?

14. How many times is $\frac{5}{7}$ contained in the following numbers?

(1) in $\frac{3}{8}$? (2) in $\frac{5}{7}$? (3) in $\frac{7}{8}$? (4) in $\frac{9}{10}$? (5) in $\frac{7}{15}$? (6) in 12? (7) in a? (8) in $\frac{a}{b}$? (9) in $\frac{x}{y}$?

15. Find the result in the following indicated divisions:

(1) $\frac{3}{5} \div \frac{2}{7}$; (2) $\frac{4}{7} \div \frac{6}{11}$; (3) $\frac{7}{9} \div \frac{6}{7}$; (4) $\frac{3}{8} \div \frac{7}{5}$; (5) $\frac{a}{b} \div \frac{c}{x}$; (6) $\frac{c}{x} \div \frac{d}{f}$.

16. Make a rule for dividing one fraction by another.

17. Apply your rule to the following:

(1) $\frac{7}{8} \div \frac{3}{5}$; (2) $\frac{6}{11} \div \frac{5}{7}$; (3) $\frac{m}{l} \div \frac{s}{t}$; (4) $\frac{p}{q} \div \frac{m}{n}$; (5) $\frac{c}{x} \div \frac{a}{f}$.

PRINCIPLE II.—*Fractions are multiplied by multiplying their numerators for the numerator of the product, and multiplying their denominators for the denominator of the product.*

PRINCIPLE III.—*One fraction is divided by another by multiplying the dividend by the inverted divisor; that is, by the reciprocal of the divisor.*

QUERIES.—(1) Why is the divisor inverted?

(2) Why is the inverted divisor multiplied by the dividend?

CHAPTER VI

USES OF THE EQUATION

§ 14. Problems

1. Divide a pole 20 ft. long into two parts so that one part shall be 4 times as long as the other.

ARITHMETIC SOLUTION

The shorter part is a certain length.

The longer part is four times this length.

The whole pole is then five times as long as the shorter part.

The pole is 20 ft. long.

The shorter part is $\frac{1}{5}$ of 20 ft, or 4 ft.

The longer part is $4 \cdot 4$ ft., or 16 ft.

Hence, the parts are 4 ft. and 16 ft. long.

ALGEBRAIC SOLUTION

Let x be the number of feet in the shorter part,
then $4x$ is the number of feet in the longer part,
and $x+4x$, or $5x=20$,

$$x=4,$$
$$4x=16.$$

Hence, the parts are 4 ft. and 16 ft. long.

2. A farmer wishes to enclose a rectangular pen with 80 ft. of wire fencing. He wishes it to be three times as long as it is wide. How long shall he make each side?

ALGEBRAIC METHOD

Let x be the number of feet in the smaller side,
then $3x$ is the number of feet in the longer side,
and $x+3x$, or $4x$ is number of feet half-way round the pen.

$$4x=40,$$
$$x=10,$$
$$3x=30.$$

Hence, the sides are 10 feet and 30 feet long.

3. James has 3 times as many cents as Charles, and 4 times as many as William. All together they have 57 cents. How many cents has each?

4. A boy sold a certain number of newspapers on Monday, twice as many on Tuesday, 10 more on Wednesday than on Monday, and 24 on Thursday. He sold 94 in the four days. How many did he sell on each day?

5. A man divides up his 160 acre farm as follows: He takes a certain number of acres for lots, 4 times as much for pasture, 4 times as much for corn as for pasture, $\frac{1}{2}$ as much for wheat as for corn, and 15 acres for meadow. How many acres does he assign to each purpose?

6. A May-pole 22 ft. high breaks into two pieces so that the top piece, hanging beside the lower piece, lacks 6 ft. of reaching the ground. How long is each piece?

7. A pony, a saddle, and a bridle together cost \$120. The bridle costs $\frac{1}{5}$ as much as the saddle, and the pony costs \$12 less than 12 times as much as the saddle. What was the cost of each?

8. A bicyclist rode a certain number of miles on Monday, $\frac{3}{4}$ as many miles on Tuesday, $\frac{5}{8}$ as many on Wednesday, $\frac{3}{4}$ as many on Thursday, as many on Friday as on Monday, and 20 miles on Saturday. On the six days he rode 152 miles. How many miles did he ride each day?

9. The area of a triangular piece of ground is 315 sq. rd. One side is 30 rods. How long is a fence at right angles to this side from the opposite corner? (Use principle below.)

PRINCIPLE.—*The area of a triangle is equal to $\frac{1}{2}$ the product of its base and altitude.*

10. If this fence divides the side (30 rods) so that one part is twice as long as the other, what are the areas of the two lots?

11. If the fence divides the side (30 rods) so that one part is five times the other, what are the areas of the two lots?

12. A local train goes at the rate of 30 miles an hour. An

express starts two hours later and goes at the rate of 50 miles an hour. In how many hours, and how far from the starting point will the second train overtake the first?

13. A book dealer has in stock twice as many Readers as Arithmetics, four times as many Readers as Histories. In all he has 70 Readers, Arithmetics, and Histories. How many of each has he?

§ 15. The Sum of the Three Angles of Any Triangle

Draw any triangle. From the vertex of one angle draw a perpendicular to the opposite side. Cut out the triangle and fold so that the vertices all meet at the foot of the perpendicular. What seems to be the sum of the three angles of the triangle? This is sometimes called *Pascal's Method.* Is

FIG. 14

the sum of all the angles formed about a point on one side of a straight line always equal to the two right angles, or 180 degrees? Fig. 14.

Draw any triangle. Lay a pencil flat along one side of the triangle and note which way it points. Revolve it about one vertex as a pivot *across the triangle* till it coincides with the next side, continue this process at the two other vertices. Through what part of a circle has the point of the pencil revolved? Does this seem to verify *Pascal's Method?*

1. Find the value of each angle of a triangle in right angles or degrees, if the first angle is twice the second, and the third is three times the first.

2. One acute angle of a right triangle is $\frac{3}{7}$ of a right angle. Find the other.

§ 16. Angles Made by Two Intersecting Straight Lines

1. One angle of two intersecting straight lines is $\frac{3}{5}$ of a right angle. Find the other three.

Fold two intersecting straight creases in a paper. Fold through the vertex and bring the upper side of one vertical angle along the upper side of the other. How do the vertical angles compare as to size? Treat the other vertical angles in the same way. State a principle about the relative size of vertical angles.

2. One of the angles formed by two intersecting straight lines is 15 degrees. Find the other three.

3. One of the angles formed by two straight lines is $\frac{1}{4}$ R. A. Find the other three.

4. The sum of one angle and its vertical angle is ten times the sum of the other two angles. Find the size of each angle.

5. The difference between one pair of vertical angles and the other is 28°. What is the value of each angle?

§ 17. The Sums of the Angles of Polygons

1. Make any quadrilateral and divide it into two triangles by drawing a diagonal. What is the sum of all the angles of the quadrilateral?

2. Make a pentagon, and draw all diagonals possible from one vertex. What is the sum of the angles of the pentagon? What is the sum of the angles of a hexagon? An octagon? An n-gon?

3. Into how many triangles is any polygon divided by the diagonals from a single vertex?

4. State a law as to the number of right angles in relation to the number of sides of a polygon?

5. Make polygons of various numbers of sides. From a point *within* each polygon draw lines to all of its vertices.

Find the sum of the angles and deduct the number of right angles formed about the point from which the lines were drawn. How many right angles does that leave for the angles of the polygon? State a law regarding the sum of the angles of any polygon.

Interpret $S=(n-2)$ times $180°$, as expressing the sum of the angles of any polygon, if n is the number of sides.

6. Using your law, find the sum of the angles of a polygon of 16 sides; of 25 sides; of 20 sides; of 39 sides.

7. Find the number of sides of the polygon in which $S=7,200°$; $S=40$ R. A.; $S=190$ R. A.

8. Find the value of *one* angle of an equiangular polygon of ten sides.

9. How many sides has an equiangular polygon in which one angle is $150°$?

10. How many sides has an equiangular polygon in which the sum of *two* angles is $270°$?

§ 18. The Exterior Angles of Regular Polygons.

An exterior angle of a polygon is formed between one side produced and the adjacent side. In Fig. 15, CAB, or EAD, is the exterior angle. Does the extension of one side or the other at any vertex, make any difference in the size of the exterior angle? Why?

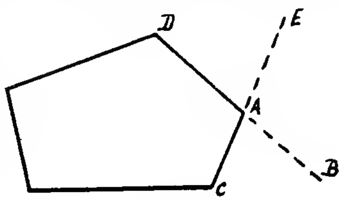

FIG. 15

1. Revolve a pencil through each exterior angle of any polygon. Through what part of a circle does the pencil point revolve? What do you conclude as to the sum of the exterior angles of any polygon?

2. How many sides has an n-gon in which the sum of the interior angles is seven times the sum of the exterior angles?

3. How many sides has an equiangular n-gon in which one exterior angle is $\frac{2}{7}$ of a R.A.

4. As the sum of the exterior and interior angles at each vertex of an n-gon is 2 R.A., if you deduct the *exterior angles* from the sum of both exterior and interior angles at all the vertices, what is left as the sum of the interior angles of an n-gon? Compare this with your previous conclusion on p. 33.

CHAPTER VII

USES OF INEQUALITIES

§ 19. Laws for Use of Expressions of Inequality

The form $4>3$ is read "four is greater than three," and the form $3<5$ is read "three is less than five."

Since $4>3$, then $4+2>3+2$, and $4-1>3-1$. Or, $4+k>3+k$, and $4-l>3-l$.

Moreover, if $a>b$, then $a+c>b+c$, and $a-c>b-c$.

These examples illustrate the facts: (1) that if equal numbers are added to unequal numbers, the sums are unequal in the same way (or *order*); and (2) if equals are subtracted from unequals the remainders are unequal in the same way (or *order*).

As $8>5$, so $2 \cdot 8>2 \cdot 5$, and $-2 \cdot 8<-2 \cdot 5$.

This example illustrates the principle that, *if unequal numbers are multiplied by the same or by equal numbers, the results are unequal in the same order, if the multiplier is positive, and the results are unequal in the opposite order if the multiplier is negative.* I. e., if $c>d$ and a be any positive number, then (1) $ac>ad$ and (2) $-ac<-ad$.

Proof.—If $c>d$, then $c-d$ is a positive number. Why? Hence $a(c-d)$ is positive (a being positive), or $ac-ad$ is positive and $ac>ad$. Why?

$-a(c-d)$ is a negative number. Why? Hence, $-ac-(-ad)$ is a negative number. Then $-ac<-ad$. Why?

Since dividing is the same as multiplying by the reciprocal of the divisor, the last principle is true for dividing unequal numbers by the same or by unequal numbers.

§ 20. Problems with Inequalities

1. Of what different whole numbers is it true that one-half of the number increased by 5 is greater that four times four-thirds of it diminished by three?

2. A factory finishes a certain number of wagons a day. If 6 more each day were finished it would make more than 240 a week. If 16 less each day were done there would be an output of less than half of 240 a week. How many were finished each day?

3. If $a>b$ and $b>c$, prove $a>c$.

4. The sum of the squares of any two different numbers is greater than twice their product.

Proof. $(a-b)^2>0$, because the square of any number is a positive number. Therefore, $a^2-2ab+b^2>0$. Adding $2ab=2ab$, the result is $a^2+b^2>2ab$.

5. Prove that the sum of the squares of any number (except 1) and its reciprocal is greater than 2.

Suggestion:

$$\left(\frac{a}{b}-\frac{b}{a}\right)^2>0. \quad \text{Why?}$$

6. Is any side of a triangle less than the sum of the other two? Prove. Greater than the difference? Prove.

7. Prove in the accompanying figure that $BD+DC<BA+AC$.

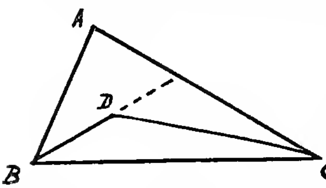

FIG. 16

8. Prove the sum of three lines drawn from a point within a triangle to the three vertices is less than the perimeter of the triangle.

9. An exterior angle of a triangle is greater than either of the interior non-adjacent angles. Prove.

10. Prove angle $D>$ angle A, Fig. 16.

CHAPTER VIII

THE OPERATIONS OF ARITHMETIC ABBREVIATED

§ 21. General Arithmetic

1. Denote the minuend by m, the subtrahend by s, and difference by d. Show by an equation the relation of these numbers.

Answer: Equation, $d = m - s$. Interpretation: "difference equals minuend minus subtrahend."

2. The multiplicand is M, the multiplier m, and the product P. Express their relation by an equation and translate the equation into words.

3. Add s to both sides of the equation $d = m - s$ and translate the result. Show that the result is a rule for checking, or testing, subtraction.

NOTE.—Observe that $s - s$, or $-s + s = 0$.

4. Divide both sides of $P = M \cdot m$, or $P = Mm$, (a) by m and interpret (translate into words) the result, (b) by M, and interpret the result.

5. Show by an equation the relation between the dividend, D, divisor, d, and quotient, q. Interpret the equation.

6. Show by an equation the relation of the dividend D, divisor d, quotient q, and remainder r, and interpret the equation.

Express by equations the relations of the numbers in the following problems:

7. A boy has m marbles and buys b more. He then has M marbles.

8. There are b boys and g girls in a class of p pupils.

9. A boy earns c cents a day for d days. He then has C cents.

10. A rectangular flower bed is l feet long and contains s square feet. The width is w feet.

11. A bicyclist rides M miles, which is r miles more than t times m miles.

12. The sum of the fractions $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ is S.

13. The difference of the fractions $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ is D $\left(\frac{n_1}{d_1}\text{ is the minuend-fraction}\right)$.

14. The product of the fractions $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$ is P.

15. The base of a rectangle is b ft. and the altitude is a ft. The area is R sq. ft.

16. Each side of a square is s ft. and the area is A sq. ft.

17. The value of a load of corn of b bu. at c ct. per bu. is D cents; D dollars.

18. The quotient of the fraction $\frac{n_1}{d_1}$ divided by $\frac{n_2}{d_2}$ is Q.

19. A decimal fraction has three units in tenths place, and 5 units in hundredths place, and its value is v.

Answer: $v=\frac{3}{10}+\frac{5}{100}$.

20. A decimal fraction has t units in tenths and h units in hundredths place and its value is v.

21. A decimal has a units in tenths place and b units in hundredths place and c units in thousandths place. Its value is v.

22. A mixed number has a_1 units in hundredths place, a_2 units in tenths place, a_3 in units place, a_4 in tens place, and a_5 in hundreds place and a_6 in thousands place. Its value is v.

23. The base is b bushels, the rate r per cent., and the percentage p bu.

24. The base is b lb., the rate r per cent., and the percentage p lb.

25. The base is b, the rate r, and the percentage p.

§ 22. Laws of Percentage and Interest

1. Divide both sides of the equation $p=\frac{b \cdot r}{100}$ by b and interpret your result.

2. Multiply both sides of the equation $p=\frac{br}{100}$ by 100, then divide both sides by r, and interpret.

3. Multiply both sides of the equation $p=\frac{br}{100}$ by 100, then divide both sides by b and interpret your result.

4. Express by an equation the relations involved if the interest is \$$i$, the principle \$$p$, the rate $r\%$ and the time t yrs.

5. Divide both sides of the equation $i=\frac{prt}{100}$ by rt, and then multiply by 100, and interpret.

6. Divide both sides of the equation $i=\frac{prt}{100}$ by pr, then multiply both sides by 100, and interpret.

7. Multiply both sides of the equation $i=\frac{prt}{100}$ by 100, then divide by pt and interpret.

QUERY.—What is the product of any fraction by a number equal to its own denominator?

CHAPTER IX

THE EVALUATION OF EXPRESSIONS

§ 23. The Circle and Sphere

1. The circumference of a circle is equal to $\frac{22}{7}$ of the diameter; i. e., $C = \pi d$, where $\pi = \frac{22}{7}$.

Find C, if (1) $d = 21$ ft.;
(2) $d = 7$ ft.;
(3) $d = \frac{1}{2}$ foot;
Find d, if (1) $c = 88$ ft.;
(2) $c = 66$ ft.;
(3) $c = 16$ feet.

2. If r stands for the radius of a circle and C for its circumference, from problem 1, show that $C = 2\pi r$.

Find C, if (1) $r = 3$ ft.;
(2) $r = 6$ ft.;
(3) $r = 18$ feet.

3. The area of a circle equals $\frac{22}{7}$ times the square of the radius; i. e., $A = \pi r^2$, where $\pi = \frac{22}{7}$.

Find A, if (1) $r = 3$ ft.;
(2) $r = 7$ ft.;
(3) $r = 21$ feet.
Find r, if (1) $A = 154$ sq. ft.;
(2) $A = 220$ sq. ft.;
(3) $A = 86.4$ square feet.

4. The volume of a sphere equals $\frac{4}{3}\pi$ times the cube of the radius; i. e., $V = \frac{4}{3}\pi r^3$, where $\pi = \frac{22}{7}$.

Find V, if (1) $r = \frac{7}{2}$ ft.;
(2) $r = 11$ ft.;
(3) $r = 2$ feet.

§ 24. Motion and Mensuration

1. The distance traversed by a moving body is equal to the rate multiplied by the time; i.e., $D = rt$.

Find D, if (1) $r = 30$ ft. per second, and $t = 5$ seconds;
(2) $r = 5$ mi. per hour, and $t = 17$ hours.

2. The area of a rectangle is equal to the product of the base and the altitude; i. e., $A = ba$.

Find A, if (1) $b = 13$ ft., and $a = 24$ ft.;
(2) $b = 10.2$ in., and $a = 3.5$ inches.

3. The area of a triangle is equal to $\frac{1}{2}$ the product of the base by the altitude; i. e., $A = \frac{1}{2}ba$.

Find A, if (1) $b = 12$ ft., and $a = 16$ ft.;
(2) $b = 8.2$ rd., and $a = 7.78$ rods.

4. The area of a parallelogram is equal to the product of the base by the altitude; i. e., $A = ba$.

Find A, if (1) $b = 28$, and $a = 19$;
(2) $b = 16.3$, and $a = 14.6$

5. Find the numerical value of each of the following expressions, when $a = 5$, $b = 3$, $c = 10$, $m = 4$, $n = 1$:

(1) $5cm^2$;
(2) $3a^3c$;
(3) $13b^2cn^2$;
(4) $n^5 + 1$;
(5) $5abc + 3m^2n^3$;
(6) $\dfrac{c + 2m}{c - 2m}$;
(7) $a^2 + b^2 + c^2 + m^2 + n^2$;
(8) $2ab + 4bc + 5cm^2$;
(9) $a^3 - b^3 + c^3$;
(10) $3abcmn$.

6. A stone, falling from rest, goes in any given time 16 ft. multiplied by the square of the number of seconds it has fallen; i. e., $s = 16t^2$.

Find s, if (1) $t = 4$ seconds;
(2) $t = 11.5$ seconds.

7. A stone thrown downward goes in any given time 16 ft. multiplied by the square of the number of seconds it

has fallen, plus the product of the velocity with which it is thrown and the number of seconds fallen; i. e., $s=16\,t^2+vt$.

Find s, if (1) $t=12$ seconds, and $v=3$ ft. per second;
(2) $t=8$ seconds, and $v=7$ ft. per second.

8. The time, t, taken for a pendulum to make a single vibration equals $\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum in feet, g is 32, and π is $\frac{22}{7}$.

Find t, if (1) $l=8$ ft.;
(2) $l=\frac{392}{121}$ feet.

Find l, if (1) $t=1$ second;
(2) $t=4$ seconds.

9. The area, A, of a triangle is equal to

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, c represent the lengths of the sides and $s=\frac{1}{2}(a+b+c)$.

Find A, if (1) $a=3$, $b=4$, $c=5$;
(2) $a=5$, $b=12$, $c=13$.

10. $H=\dfrac{lf}{550S}$. Find H, if (1) $l=11$, $f=5$, $S=2$;
(2) $l=10$, $f=33$, $S=6$.

11. $R=\dfrac{rr'}{r+r'}$. Find R, if (1) $r=11.5$, $r'=6\cdot5$;
(2) $r=13$, $r'=15$.

12. $E=\dfrac{MV^2}{2}$. Find E, if (1) $M=12$, $V=5$;
(2) $M=11$, $V=9$.

Find M, if (1) $E=8$, $V=4$;
(2) $E=50$, $V=5$.

13. $V=\frac{1}{3}h(B+b+\sqrt{Bb})$.

Find V, if (1) $h=9$, $B=16$, $b=4$;
(2) $h=6$, $B=16$, $b=9$.

14. Find the numerical value of each of the following expressions, when $a=8$, $b=6$, $c=1$, $x=b$, $y=4$:

(1) $\frac{5}{6}a-\frac{1}{8}b^3+\frac{7}{8}y^2$;

(2) $\frac{3a^2b}{cxy^2}-\frac{5y}{a}$;

(3) $\sqrt[3]{\frac{6cy^3}{b}}+2\sqrt{\frac{3a^3}{4b^3}}$;

(4) $\sqrt{bxy}-\frac{1}{8}b^2+\frac{8x^2}{b^2y}$.

CHAPTER X

GEOMETRIC REPRESENTATION OF QUANTITY

§ 25. Drawing to Scale

1. How would you draw a map of a rectangular field 50 ft. long and 40 ft. wide on a piece of paper 8 in. by 10 in.? What relation would exist between the field and the map?

2. Draw a line 3 inches long, and let it represent a distance of 48 ft. What distance would be represented by 1 inch? by 2 inches? by 6 inches? by $1\frac{1}{2}$ inches? by $2\frac{3}{8}$ inches? by $1\frac{1}{16}$ inches?

NOTE.—In the above example the drawing is said to be made *to a scale of 1 inch to 16 feet.*

3. Draw to the same scale: 8 ft.; 12 ft.; 24 ft.; 28 feet.

4. If a line 5 inches long is taken to represent a distance of 75 miles, what scale is used?

5. Make a drawing to the scale of $\frac{1}{4}$ inch to 1 rod ($\frac{1}{4}'' =$ 1 rd.) of a rectangular field 16 rods long and 12 rods wide.

6. Fig. 17 represents a seven-sided field, and Fig. 18 is a scale drawing of Fig. 17. The distances from O to the several

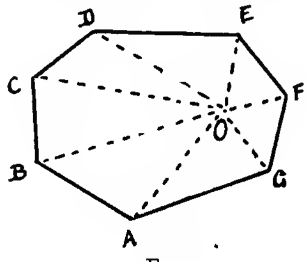

FIG. 17

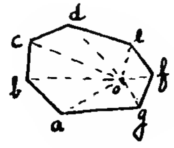

FIG. 18

corners of the field represented in Fig. 17 are: to A, 678 ft.; to B, 612 ft.; to C, 683 ft.; to D, 738 ft.; to E, 698 ft.; to F, 625 ft.; to G, 679 ft.

If the scale of Fig. 18 is $1'' = 200'$ what is the length in inches, to two decimal places, of the lines oa, ob, oc, od, oe, of, og?

7. If, in Fig. 18, ab=4.12 inches, bc=4.12 inches, cd=2.86 inches, de=2.75 inches, ef=5.86 inches, fg=6.18 inches, and ga=5.98 inches, what are the lengths in feet of the corresponding sides AB, BC, etc., of the field?

8. If the area of triangle oab (Fig. 18) is 4.94 sq. in., what would be the area of triangle OAB (Fig. 17), if the scale is 1 inch to 200 feet? Answer the same question using a scale of 1 inch to 100 feet.

9. Draw to a scale of $\frac{1}{4}'' = 10$ rd. a garden plot (from the data of Fig. 19). Use a protractor to draw the angle A.

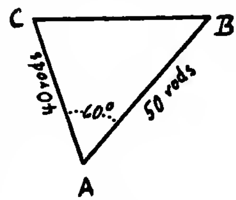

Fig. 19

How many rods long is the side BC?

How many degrees in the angle B? In angle C? What is the sum of the three angles of the triangle ABC? What do you infer from this?

10. A railroad surveyor wishes to measure across the swamp AB (Fig. 20). He measures the distance from the tree at A to a stone at C and finds it to be 150 ft. The distance from the tree at B to the stone is 165 ft. The angle* at C is 85°. Draw to a convenient scale the triangle ABC and determine from your drawing the distance in feet across the swamp.

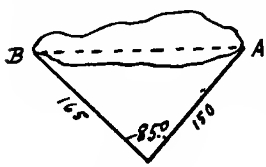

Fig. 20

11. A man wishes to measure the width, AC, of a stream (Fig. 21) without crossing it. He lays off a line, BC, on one side of the river, and measures (with his transit) angles B and C.

* Note.—Surveyors measure angles with an instrument called the transit. For rough measurements the pupil can easily construct an angle-measurer by tacking a protractor on a board. A ruler with a pin stuck in it at each end can be used as a sight.

Draw a triangle carefully from the data given in the figure, and determine the width of the river by measuring AC with a ruler.

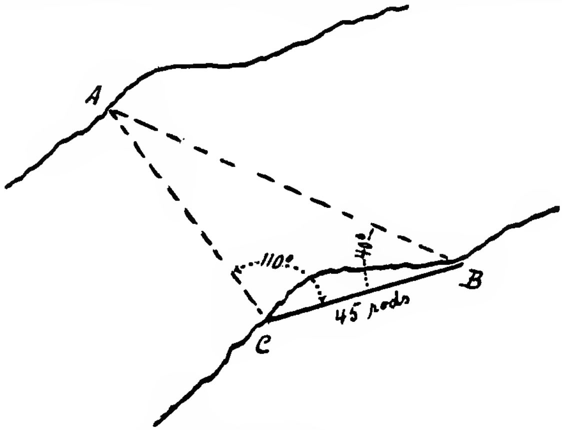

FIG. 21

12. A boy wishes to determine the height HK (Fig. 22) of a factory chimney. He places the angle-measurer first at B and then at A and measures the angles x and y. The angle-measurer lies on a box, or tripod, $3\frac{1}{2}$ ft. from the ground. A

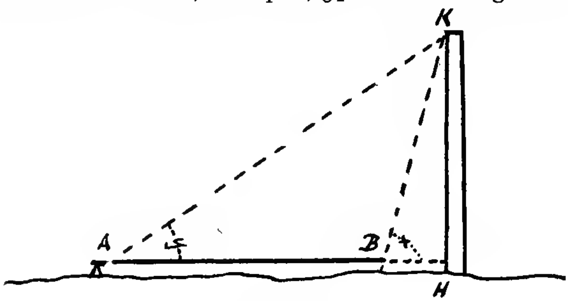

FIG. 22

and B are two points in line with the chimney, and at a convenient distance apart. Show by a drawing how the boy may determine the height, HK, of the chimney, if the ground is level, and if $x=63°$, $y=33\frac{1}{2}°$, and $AB=50$ feet.

13. Determine the height of the flag-staff on your school building by making measurements similar to those used above. For this purpose construct an angle-measurer according to the directions in the foot note on page 45.

14. Find the distance AB (Fig. 23), through some building near your school. First select some point O from which you can see both A and B. Measure angle x with the angle-measurer. Also measure OA and OB with a yard-stick, or surveyor's tape, or chain. Make a scale drawing of the triangle AOB, and measure the side AB with a ruler. How long is the actual distance AB?

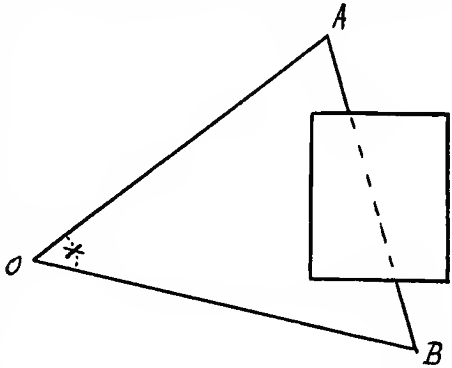

FIG. 23

15. A surveyor wants to know the distance MQ (Fig. 24)

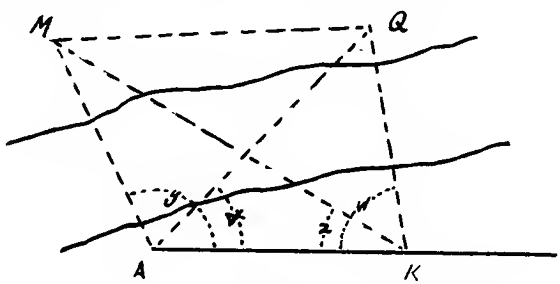

FIG. 24

between two trees on the opposite bank of a river. He places his transit first at A, measuring the angles x and y; and then at K, measuring the angles w and z. Make a scale drawing

and find the distance MQ, if AK=50 rods, angle $x=45°$, angle $y=115°$, angle $z=30°$, and angle $w=80°$.

Hint: First draw triangle AMK, and then draw triangle AQK.

16. A flag-pole 40 ft. high casts, on level ground, a shadow 60 ft. long. Draw the triangle to scale and measure the angle which shows the angle of elevation of sun at the time.

17. Draw at least three or four triangles, ABC, for each of the sets of given parts in the table below. The symbol ($\angle$) means angle.

Compasses are needed for triangles IV and VI. In the completed triangles draw heavy lines for the given sides and broken, or dotted, lines for the other sides. Indicate the given angles by an arc. Thus, for triangle III:

No.	AB	BC	CA	$\angle$A	$\angle$B	$\angle$C
I	1″		2″			
II		1″				30°
III		1″	$1\frac{1}{2}$″			60°
IV	1″	$1\frac{1}{2}$″	2″			
V	1″			15°	30°	
VI		2″	$1\frac{1}{2}$″		30°	
VII	$1\frac{1}{2}$″	1″		60°	30°	

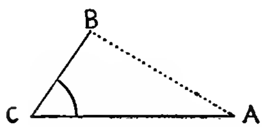

FIG. 24*a*

18. Can a triangle be drawn whose sides are 3″, 2″, and 6″? $2\frac{1}{3}$″, $4\frac{2}{3}$″, and 7″?

19. Two sides of a triangle are 4″ and 9″ in length. Between what limits must the length of the third side be?

20. In I, above, are all the triangles necessarily of the same size and shape? In II? In III? In IV? In V? In VI?

21. Can you make triangles by omitting any one of the four given parts in VII?

§ 26. Summary

Any triangle has six parts—3 sides and 3 angles. It is to be observed in drawing the triangles of the above table, that when less than three parts of a triangle are given many triangles having them can be drawn. These triangles are, however, not necessarily of either the same size or shape.

It is also to be noted that when *three* parts are given (with two exceptions to be considered later), all the triangles, which have these three given parts, are necessarily of the same shape and size. The three given parts are said to *determine* the triangle. Such triangles are really the same triangle in different positions. Any one of the triangles may be so placed upon another, that the corresponding parts of the two triangles will *coincide*, or fit exactly. They are called *equal*, or *congruent triangles*.

§ 27. Exceptional Cases

First Exception—

If the value of angle B in VII is omitted, we have an example of the first exception mentioned above.

1. Point this out in drawings of triangles having the following parts:

(1) $AB = 1\frac{1}{2}$ in., $BC = 1$ in., and $\angle A = 60°$;
(2) $AB = 3\frac{1}{2}$ in., $BC = 3$ in., and $\angle A = 60°$;
(3) $AB = 4$ in., $BC = 2\frac{1}{2}$ in., and $\angle A = 30°$.

Second Exception—

1. Draw three or four triangles in each of which the angles are respectively 35°, 65°, and 80°. Are all of those triangles necessarily of the same *size?* Do they all have the same *shape?*

2. Draw three or four triangles each of which has angles 90°, 25,° and 65°. Are all of these triangles necessarily of the same size and shape?

Triangles having the same shape are called *similar triangles*. Similar triangles are not necessarily of the same size.

§ 28. Triangles Having the Same Shape (Similar Triangles)

1. Draw a triangle with sides 2″ and 3″, respectively, and an angle of 50° included between them. First draw it actual size and then to the scale of $\frac{3}{8}'' = 1''$. Measure with a protractor the pairs of corresponding angles. Are the two triangles similar? Why?

2. These triangles though different in size have the same *shape* and have their corresponding angles equal. Notice also that each side of the smaller triangle is $\frac{3}{8}$ of the length of the corresponding side of the larger triangle. All *similar* triangles may be regarded as the same triangle drawn to different scales.

They may be regarded as the same triangle *magnified,* or *minified* to a definite scale.

3. Draw any triangle, and then draw another one similar to it, to a scale of 2 : 1. Draw another to a scale of $\frac{1}{2}$: 1.

4. Two rectangular flower-beds have the same shape, but are different in size. One is 3 ft. wide and 5 ft. long; the other is 12 ft. wide. How long is it?

5. Two books have the same shape. One is $5\frac{1}{2}$ inches wide and $7\frac{1}{2}$ inches long. The other is 15 inches long. How wide is it?

6. The top of a desk and a rectangular sheet of paper 12 in. × 18 in. have the same shape. The desk is 2 ft. wide. How long is it?

7. A city block and a lot within the block have the same shape. The lot is 100 ft. by 150 ft. and the block is 300 ft. wide. How long is it?

8. The gables of a house and of a porch have the same shape. The sides of the porch-gable are 7 ft., 7 ft. and 10 feet. The longest side of the house-gable is 25 ft. How long are the other two sides of the house-gable?

9. In two triangles of the same shape, like those of Fig. 25 (1), if $a=4$ inches, $A=12$ inches and $b=4$ inches, how long is B?

10. In triangles of the same shape, like those of Fig. 25 (2), if $a=4$ in., $A=12$ in., and $b=5$ in., how long is B? If $a=3$ in., $b=8$ in., and $B=32$ in., how long is A? If $a=x$ in., $b=8$ in., and $B=32$ in., how long is A?

11. In triangles of the same shape, like those in Fig. 25 (3), if $A=21$ in., $b=9$ in., and $B=27$ in., how long is a? If $a=5\frac{1}{2}$ in., $A=22$ in., and $B=30$ in., how long is b?

12. In triangles of the same shape, like those in Fig. 25 (4), if $a=3$ in., $A=8$ in., and $C=5$ in., how long is c? If $A=24$ in., $c=4$ in., $C=7$ in., how long is a? If $A=y$ in., $c=4$ in., and $C=7$ in., how long is a?

13. The shortest side of a triangle is 12 ft., the longest is 24 ft., and the third side is 16 feet. If a similar triangle has its shortest side 8 ft., what are the lengths of the other two sides?

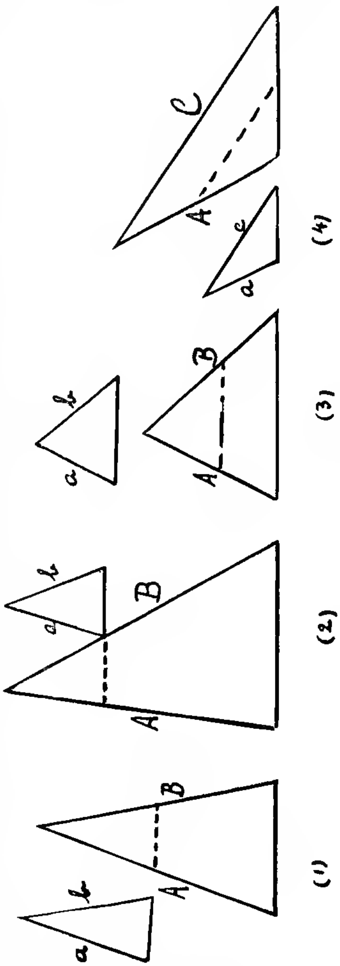

FIG. 25

14. In Fig. 26, if the stake 3 ft. high casts a shadow 8 ft. long, and the tree, at the same time, casts a shadow 80 ft. long, how high is the tree?

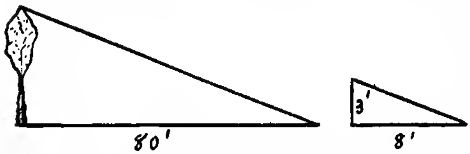

FIG. 26.

15. Measure the height of some tree, building, or flag-pole, by the method of shadows as in problem 14.

16. How long is x in Fig. 27? Are triangles OAB and OHK similar? Give reason for answer.

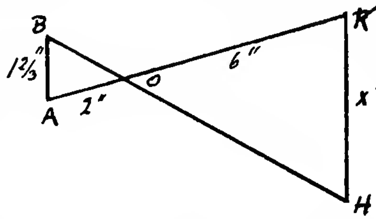

FIG. 27

17. A boy holds a pencil, AB (Fig. 28), $2\frac{1}{2}$ ft. from his eye, so that it covers a flag-pole 360 ft. away. To make the triangles EAB and EFK similar, how must the pencil be held? If the pencil is 7 in. long, how high is the pole?

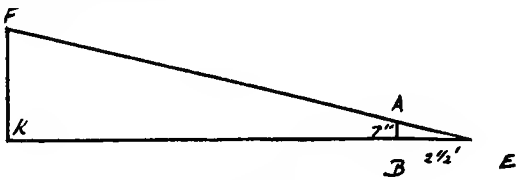

FIG. 28

18. A lumberman who is 5 ft. tall wishes to find a tree 60 ft. to the first limbs. He drives a stake in the ground and places his feet against it as in Fig. 29. If the stake is 4 ft. high, how far must it be placed from the foot of the tree, that he may determine whether or not the trunk is 60 ft. to the limbs?

19. In Fig. 30, the letters a and b denote the same numbers

throughout. How do the areas of triangles I, II, and III compare? I and IV? II and VI? III and V? III and VI? IV and VIII? I and VIII? III and IX? IX and X? VII and X? III and X?

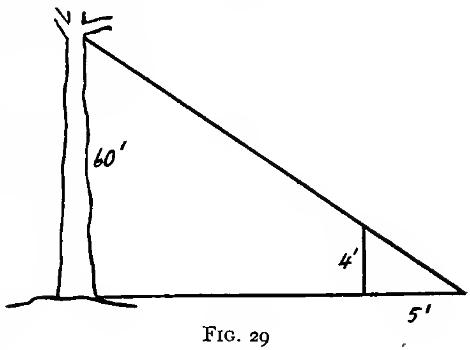

FIG. 29

20. If the altitude and base of a triangle are 4′ and 15′

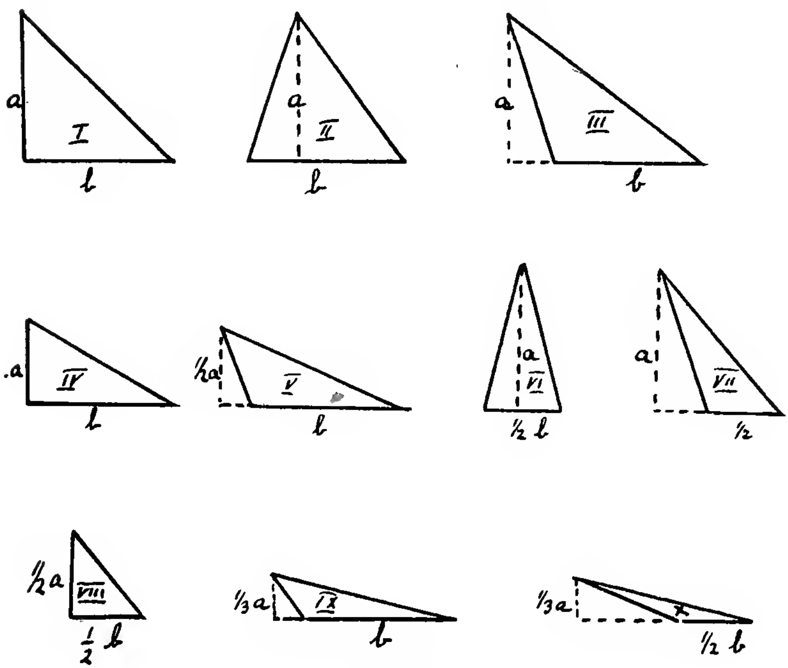

FIG. 30

respectively, what are the dimensions of other triangles of $\frac{1}{2}$ the area? of $\frac{1}{3}$ the area? of $\frac{1}{6}$ the area?

21. If the altitude and base of a rectangle are 6″ and 8″ respectively, what are the altitude and base of triangles whose areas are (1) equal to the area of the rectangle? (2) twice the area of the rectangle? (3) $\frac{1}{2}$ the area of the rectangle?

22. Answer the same questions when the altitude and base of the rectangle are a and b inches, respectively.

23. A triangle and a rectangle have equal bases and are equal in area. How do their altitudes compare?

24. If the dimensions of a rectangular parallelopiped are $4' \times 15' \times 25'$, what are the dimensions of other rectangular parallelopipeds having $\frac{1}{2}$ the volume? $\frac{1}{3}$ the volume? $\frac{1}{5}$ the volume? $\frac{1}{6}$ the volume? $\frac{1}{10}$ the volume?

25. How would you obtain the area of the field polygon in Fig. 17?

CHAPTER XI

EQUATION APPLIED TO SIMPLE PROBLEMS ON BEAMS

§ 29. Common Uses of Forces

In this chapter some practical problems arising out of the common uses of forces will be solved by means of the equation. It is necessary first to discover a law of these forces.

Arrange an apparatus like that in Fig. 31, consisting of a light wooden bar 10″ long, provided with pegs (small nails). A cord passes over the pulley, K, and is attached at one end to the middle, M, of the bar, and at the other, to a light scale-pan, S. Sufficient weight is placed in the pan, S, to hold the bar in balance. If, now, weights are attached to the various pegs on the bar, and other weights are attached to the bottom of the pan, S, the bar will be acted upon by forces of two kinds. The weights on the pegs will pull downward on the bar, while those attached to the pan will pull upward on the bar by the aid of the cord and pulley. To avoid confusion, an upward pulling force will be called positive and a plus (+) sign will be written before the number showing its *strength*. On the other hand, forces pulling downward on the bar will be called negative and a minus (−) sign will be written before the magnitude number.

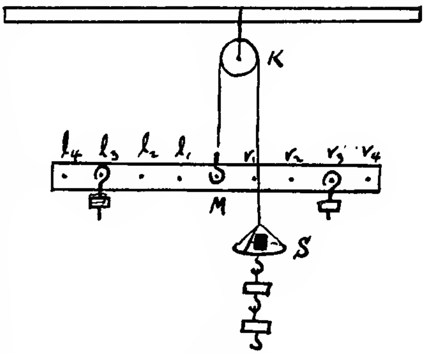

Fig. 31

§ 30. Experiments

1. Putting a weight, x, at l_3, and an equal weight, x, at r_3, it will be found that two weights, each equal to x, hung to the pan, S, will hold the apparatus in balance. Beginning on the left, record the relations for balance thus:

$$(1) \quad -x+2x-x=0.$$

2. With two weights, each equal to x at l_3, and two weights, each equal to x, at r_3, 4 weights, each equal to x, must be put at S for balance. But 3 or 5 weights, x, at S will be found not to balance. Make the record and interpretation thus:

Record	*Interpretation*
(1) $-2x+4x-2x=0$	Balance
(2) $-2x+3x-2x=-x$	Movement downward
(3) $-2x+5x-2x=+x$	Movement upward.

3. If 2 weights, x, be put at l_3, 1 weight, x, at l_2, 1 weight, x, at r_2, and 2 weights, x, at r_3, 6 weights, each equal to x, at S will balance the apparatus, but neither 5 nor 7 weights, x, will balance it. The results will run thus:

Record	*Interpretation*
(1) $-2x-x+6x-x-2x=0$	Balance
(2) $-2x-x+5x-2x-x=-x$	Bar moves downward
(3) $-2x-x+7x-x-2x=+x$	Bar moves upward.

4. If 3 weights, x, are placed at l_2 and 2 weights, x, at r_3, it will be found that 5 weights, x, at S will give balance while neither 4 nor 3 weights, x, at S will do so.

Record as above.

5. With $2x$ at l_1, $3x$ at l_2, $2x$ at r_3, and $2x$ at r_4, the following weights were tried successively at S: $7x$; $8x$; $11x$; $10x$; $9x$.

Write the equation for each case and interpret it.

6. Record the equations for these loadings: x at l_4, $2x$ at l_3, $3x$ at l_2, $4x$ at r_1, $4x$ at r_3, and the following weights, in turn, on the pan: $10x$; $12x$; $13x$; $14x$; $16x$. Interpret the equation in each case.

7. Write down the appropriate equations and state the

results as shown by a beam for each loading of the following table:

No.	l_4	l_3	l_2	l_1	S	r_1	r_2	r_3	r_4
I	x	x	x	x	$8x$	x	x	x	x
II	x	x	0	0	$3x$	0	0	x	x
III	y	0	0	0	$2y$	0	0	0	y
IV	y	0	0	y	$6y$	y	0	0	y
V	$2y$	0	$3y$	0	$15y$	$3y$	0	$2y$	0
VI	$2y$	y	$3y$	y	$18y$	$9y$	0	$3y$	0
VII	0	x	y	0	$2x+2y$	0	x	y	0
VIII	$3a$	a	$3a$	0	$25a$	$18a$	0	a	0

8. Use the same bar, supported as shown, and balance it. If now a weight, x, is placed at both L and R, two

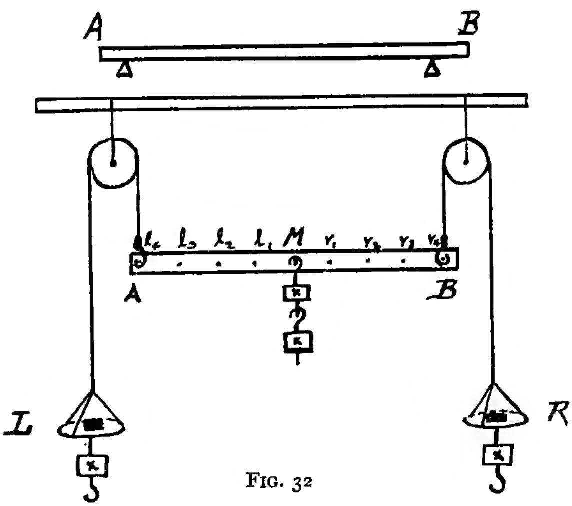

FIG. 32

weights, each equal to x at M balance the bar. The results are:

Record	*Interpretation*
$+x-2x+x=0$	Balance.

Try the same weights at L and R and $3x$ at M; x at M.

9. Eight weights, x, at l_1, 3 weights, x, at R, and 5 at L will balance the bar. If the 8 weights, x, are at r_1, then 3 weights, x, at L and 5 weights, x, at R are needed for balance. The results are:

	Record	*Interpretation*
(1)	$+5x-8x+3x=0$	?
(2)	$+3x-8x+5x=0$.	?

10. Write out the equations and state whether or not there is balance for these experiments:

No.	L	l_4	l_3	l_2	l_1	M	r_1	r_2	r_3	r_4	R
I	$6x$	0	0	$8x$	0	0	0	0	0	0	$2x$
II	$8x$	0	0	$8x$	0	$4x$	0	0	0	0	$4x$
III	$8x$	x	0	$8x$	0	$4x$	0	0	0	x	$5x$
IV	$9x$	x	0	$8x$	0	$4x$	0	0	0	x	$5x$
V	$7x$	0	$8x$	0	0	0	0	0	0	0	x
VI	x	0	0	0	0	0	0	0	$8x$	0	$7x$
VII	$2x$	x	x	0	0	0	0	0	x	x	$9x$
VIII	$3x$	x	x	0	0	$2x$	0	0	x	x	$9x$
IX	$4x$	x	x	0	0	$2x$	0	0	x	x	$10x$

When a bar is supported in two places (A and B, Fig. 32), it is called a *beam*.

In all these problems what is the test as to whether the bar, or beam, balances?

It will be readily seen that in each case of balance with either the beam, or the bar, *the sum of all the forces pulling upward must be equal to the sum of all the forces pulling downward.* This is the same as saying *that the sum of all the positive forces is equal to the sum of all the negative forces, for balance.*

A sum such as $5x-8x+3x$, which is made up, partly of positive and partly of negative numbers, is called an algebraic sum. The left sides of all the equations we have been using, in which it is required to combine positive and negative numbers into a single sum, are examples of algebraic sums.

Observe that in all these problems the forces are parallel to each other.

§31. First Law of Parallel Forces

We may now state the first law of force.

Law of Parallel Forces: *The algebraic sum of all the forces acting upon the material object (bar, or beam) must equal zero, for balance.*

We shall refer to this law as Law I of Forces.

§ 32. Practical Problems

1. A basket weighing 56 lb. (Fig. 33) is carried by two boys who lift at the ends of a light stick, AB, the basket being borne at the middle point, M. How much does each boy lift?

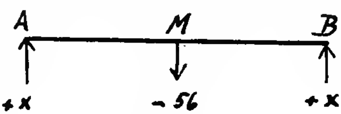

Fig. 33

Solution.—Notice that the stick is in balance under the action of three parallel forces, one pulling downward and two equal forces pulling upward. Letting x denote the number of pounds lifted by each boy, and using Law I of Forces, the equation for balance is

$$+x-56+x=0;$$

or, by collecting terms $$2x-56=0.$$

Now add 56 to both sides of the equation, thus,—

$$\begin{aligned} 2x-56&=0\\ 56&=56\\ \hline \text{and we have}\qquad 2x&=56\,;\\ \text{whence,}\qquad x&=28\,. \end{aligned}$$

Result: Each boy lifts 28 pounds.

2. Two boys, one lifting at each end, carry a ten-foot log, weighing 12 lb. per foot of length. How much must each boy lift?

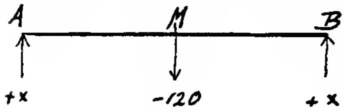

FIG. 34

SOLUTION.—Notice that the entire weight of the log may be considered as concentrated at the middle. We have then what amounts to a light bar, balanced under three parallel forces and we write:

$$\begin{aligned} +x-120+x&=0\,,\\ \text{or}\qquad 2x-120&=0\,.\\ \text{Adding 120 to both sides,}\qquad 2x&=120\,;\\ \text{whence,}\qquad x&=60\,. \end{aligned}$$

Result: Each boy must lift 60 pounds.

3. Two men, one lifting at each end of an iron bar weighing 20 lb., carry a load of 150 lb. at the middle of the bar. How many pounds does each man carry?

NOTE.—The bar being supposed to have all its weight concentrated at its middle point, we may think of the bar as being in balance under the action of four parallel forces. Draw a sketch and indicate by arrows where and how the forces act.

$$\begin{aligned} \text{The equation is then:}\qquad +x-20-150+x&=0\,,\\ \text{or combining}\qquad +2x-170&=0\,;\\ \text{adding 170 to both sides,}\qquad 2x&=170\,,\\ \text{whence}\qquad x&=85\,. \end{aligned}$$

Result: Each man lifts 85 pounds.

4. By the aid of a set of whiffle-trees (Fig. 35) two horses draw a load of 650 pounds. Show that if the force drawing backward (to the left) on the double-tree is negative, and the forces drawing forward are positive, we may find the pull of each of the four traces by this equation:

$$+2x-650+2x=0.$$

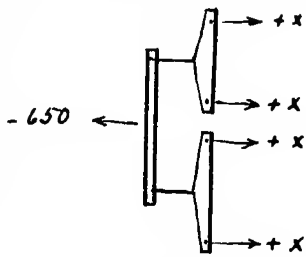

FIG. 35

Find the number of pounds, x, of pull on each trace.

5. Two boys slip a stick under one end of a 160 lb. log, and a man lifts at the other end of the log. If the boys' end of the log is on the middle of the stick, for balance how much does the man lift? How much does each of the boys lift? Show that the equation is: $+x+x-160+2x=0$. What does x stand for in the equation?

6. With two hand-spikes (AB and CD) four men, lifting equally at A, B, C, and D, respectively, carry a log weighing 600 pounds. Find by means of the equation the amount lifted by each man.

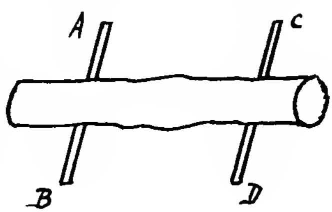

FIG. 36

§ 33. Turning-Tendencies (Leverages)

A light bar (Fig. 37) supplied with equally spaced pegs, is balanced about its middle point, M. With a number of equal weights, y, the following experiments are performed.

1. Hang a weight of $2y$ on the peg l_1. This weight tends to turn the left end of the bar downward. How much weight must be attached to the hook, H, to balance this turning-tendency? Now hang the same weight, $2y$ on peg l_2, and meas-

ure its downward turning-tendency by attaching to the hook, H, a weight sufficient to balance the bar.

2. In a similar manner find the downward turning-tendency caused by the weight $2y$ on the peg l_3; on l_4; on l_5.

3. Using a weight of $3y$, find its downward turning-tendency when placed on peg l_1; on l_2; on l_3; on l_4; on l_5.

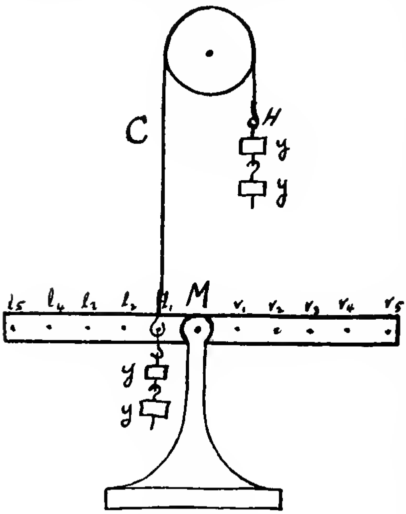

FIG. 37

4. Perform Experiment 3, using a weight $4y$.

From Experiments 1, 2, 3, and 4, it is clear that when any weight (as $2y$) is hung on peg l_5, the turning-tendency caused by this weight ($2y$) is five times as great as when the same weight ($2y$) is hung on peg l_1; on l_4 its turning-tendency is four times as great as on l_1; on l_3 it is three times, and on l_2 it is two times as great as on l_1.

The same facts hold when $3y$, $4y$, or any other weight is used. In other words, with an apparatus like that in Fig. 37, *the turning-tendency, or leverage, caused by any weight is measured by the product of that weight and the distance from the turning point, M, to the peg where the weight hangs.*

5. What is the turning-tendency caused by a weight of $7y$ hung on peg l_5? on l_3? a weight of $9y$ on l_4? on l_1?

6. If you had a bar long enough to have 12 pegs on each side of M, what would be the turning-tendency caused by a weight $8y$ on peg l_{12}? $7y$ on l_9? y on l_{11}?

7. Attach the cord, C (Fig 37), to the peg r_1 and perform experiments 1–4 on the right side of the bar.

In experiment 7, it is seen that the same facts are true on the right side as on the left, but the bar turns, or tends to turn, in the *opposite direction.* To avoid confusion, some simple method of distinguishing between these two directions of turning is desired.

Suppose a watch laid upon the page of the book with its face up. When the bar turns, or tends to turn *with* the hands of the watch, the turning-tendency will be called *negative* and designated $-$; if it turns, or tends to turn, *against* (opposite to) the watch-hands, the turning-tendency will be called *positive* and designated $+$.

In all the experiments thus far performed the weights which caused the bar to turn were downward-pulling weights, or forces. But it is evident that by arranging an apparatus as in Fig. 38, below, forces can also be made to pull upward. As before (see p. 55) we shall designate downward pulling weights or forces by $-$, and upward pulling forces by $+$.

The distance from the turning-point, M, to the peg where the weight, or force, acts will be called the *lever-arm,* or *arm,* simply, of the force. Lever-arms measured from the turning-point toward the right will be marked $+$; those toward the left, $-$.

For example, if the distance from M to peg r_1 be represented by $+x$ inches, then the distance from M to the peg r_4 will be represented by $+4x$ inches; from M to l_3, by $-3x$ inches, and so on.

If a force of $-2y$ acts on peg l_3 its turning-tendency, or leverage, would be the product of $-2y$ and $-3x$, or $6xy$. Since the bar tends to turn *against* the watch hands, the turning-tendency is written $+6xy$. If a force of $-2y$ acts on r_3, its turning-tendency would be the product of $-2y$ and $+3x$,

or $-6xy$, since the bar in this case tends to turn *with* the watch-hands.

8. In the following twelve experiments write out the leverages, or turning-tendencies, for the forces and arms indicated.

	Forces	Arms	Leverages		Forces	Arms	Leverages
I	$-3y$	$-2x$		VII	$-6y$	$+x$	
II	$-4y$	$+2x$		VIII	$-8y$	$-x$	
III	$-7y$	$-3x$		IX	$-3y$	$+7x$	
IV	$-0y$	$-4x$		X	$-4y$	$-6x$	
V	$-y$	$+5x$		XI	$-3y$	$-2x$	
VI	$-8y$	$+9x$		XII	$-6y$	$-12x$	

9. If loadings I and VII in the above tables were on the apparatus at the same time, would the bar balance or turn? If it turns, in what direction would it turn? Answer similar questions for II and VIII; V and IX; VI and XII.

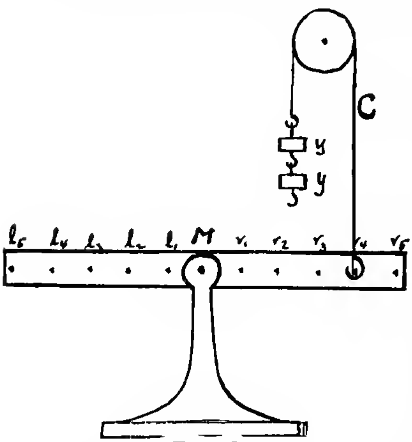

FIG. 38

By the apparatus in Fig. 38 forces can be made to pull upward on either side of the bar. If a force of $2y$ pulls upward on peg r_4 its turning-tendency is the product of $+2y$ and $+4x$, or $8xy$, and since the bar tends to turn *against* the watch-hands, the turning-tendency

is written $+8xy$. On peg l_4 the turning-tendency would be the product of $+2y$ and $-4x$, or $-8xy$, since in this case the bar would tend to turn *with* the watch-hands.

10. In the following eight experiments write out the leverages for the indicated arms and forces:

	Forces	Arms	Leverages		Forces	Arms	Leverages
I	$+3y$	$-2x$		V	$+6y$	$+x$	
II	$+7y$	$-3x$		VI	$+3y$	$+7x$	
III	$+y$	$+3x$		VII	$+3y$	$-2x$	
IV	$+8y$	$+9x$		VIII	$+6y$	$-12x$	

11. If III and VII in the above tables are on the apparatus at the same time, will the bar balance or turn? If it turns in which direction will it turn? Answer similar questions for I and V; II and VII; IV and VII.

In problem 11, if the turning-tendencies for I and V are added the *total* turning-tendency is $-6xy+6xy=0$, which says in mathematical language that the bar does not turn. If the turning-tendencies for II and VII are added, the *total* turning-tendency will be $+5xy-6xy=-1xy$, which shows that the bar does not balance, but turns in the negative direction. If two or more forces are acting on the bar at the same time, the total turning-tendency is found by adding the separate turning-tendencies. If the algebraic sum is 0 the bar balances. If the sum is not 0 the bar turns in the direction indicated by the sign of the sum.

12. In each of the following loadings find the total turning-tendency and interpret it: $+3y$ at l_5 and $-3y$ at l_5; $+2y$ at l_3 and $-3y$ at l_2; $+3y$ at r_2 and $-2y$ at r_3; $+4y$ at l_1 and $-y$ at l_4; $+4y$ at r_1 and $-y$ at r_4; $+2y$ at l_3, $-3y$ at r_3, and $-12y$ at l_1; $-2y$ at r_3, $+2y$ at l_2, and $+5y$ at r_2.

§ 34. Second Law of Parallel Forces

From the experiments and problems just given the second important law of forces becomes clear; viz.:

LAW OF TURNING-TENDENCIES OR LEVERAGES.—*For balance, the algebraic sum of all the turning-tendencies must equal zero.*

We shall refer to this as Law II of Forces.

This law is general and applies to heavy bodies as well as to the light bar used in the experiments.

By the aid of the two laws of force, now derived, we may solve a variety of practical problems.

§ 35. Practical Problems

1. A basket weighing 84 lb. hangs on a stick 6 ft. long, at a point 2 ft. from the middle, while it is being carried by two boys, one at each end. How much does each boy lift?

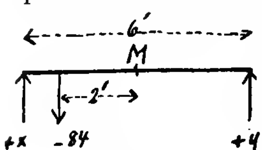

FIG. 39

SOLUTION.—Let x denote the weight carried by the boy at the short end and y denote that borne by the other boy. Suppose M to denote the middle point of the stick. Then, calling lever-arms measured from M toward the right, positive; and toward the left, negative, by the first law of forces

$$+x+y-84=0.$$

By the second law of forces

$$(+x)(-3)+(-84)(-2)+(+y)(+3)=\quad 0.$$

Simplifying these equations we have (1) $x+y=\quad 84$;

and (2) $-3x+3y=-168$.

These equations may be solved thus:

Multiply both sides of the

first by 3 and get $\quad 3x+3y=\quad 252$.

Add this equation to equation (2), thus:

$$\begin{array}{r} -3x+3y=-168 \\ 3x+3y=\ \ 252 \\ \hline 6y=+\ 84. \end{array}$$

Therefore (3) $y=+\ 14.$

But (1) $x+y=84;$

or, by subtracting equation (3) from equation (1), member from member— $x=84-14;$

that is, $x=70.$

Therefore the boy at the short end carried 70 and the other 14 pounds.

Check: (1) $70+14=\ 84,$

(2) $-210+42=-168.$

2. Solve a problem like problem 1, supposing the basket to weigh 60 lb. and to hang at a point 2 ft. from the left end of the stick; 4 ft. from the left end; 5 ft. from the left end.

3. Two men lifting at the ends of a stick 8 ft. long raise a certain weight. What is the weight, and at what point does it hang, if one man lifts 25 lb., and the other 75 pounds?

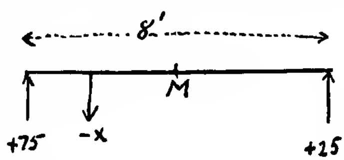

FIG. 40

SOLUTION.—The equations are:

From Law I, $+25-W+75=0$, or $W=100.$

Let x be the number of feet to the left of the mid-point of the bar.

From Law II,

$$(+75)(-4)+(-W)(-x)+(+25)(+4)=0,$$

or,

$$Wx=+200,$$
$$100x=+200,$$
$$x=+2.$$

The required weight is 100 lb. and it hangs 2 ft. from the

middle toward the one who lifts 75 pounds. Check: $+25-100+75=0$.

4. Suppose a bar 10 ft. long, weighing 30 lb., is used by two men, one grasping it at each end, to carry a load of 170 lb. How many pounds must each man carry, if the load is attached 2 ft. from the left end?

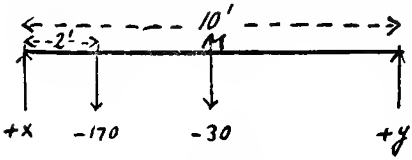

FIG. 41

NOTE.—The weight of the bar itself may be treated as a load of 30 lb. hanging to the bar at its middle point.

SUGGESTION.—Counting lever arms from the middle point of the bar the equations are:

$$(1)\quad +x-170-30+y=0,$$
$$(2)\quad -5x+510-0+5y=0.$$

NOTE.—The zero term in (2) arises from the leverage of the weight of the bar, which is $30\cdot 0$, zero being the lever-arm. But $30\cdot 0=0$, for manifestly, a weight hanging at the middle point can have no tendency to turn the bar around this point, or, what amounts to the same thing, its turning-tendency about this point equals zero.

5. A stone slab, S, weighing 2,400 lb., rests with its edge on a point, B, 6 in. from the fulcrum, F, of a crowbar, FA, 6 ft. long. How many pounds of force must a man at A exert to raise the slab?

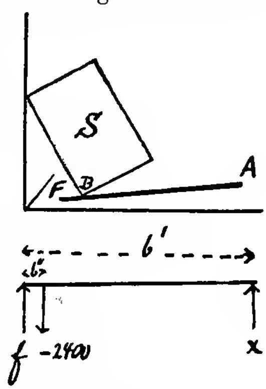

FIG. 42

NOTE.—When the stone is being supported, the bar, FA, is in balance under three forces, viz., the upward pressure of the supporting block against

the bar, the downward pressure of the stone at B and the upward pressure of the man's hand at A.

Show that the equations are:

$$(1) \quad +f-2{,}400+x=0, \qquad \text{or } f+x=2{,}400;$$
$$(2) \quad -3f+6{,}000+3x=0, \quad \text{or} -3f+3x=-6{,}000.$$

The solution of this problem enables us to find both x and f. In many such problems the force exerted by the support enters, though it rarely needs computing because the support can easily be made sufficiently strong. We shall now solve the problem without computing f. It is clear that if the bar or other body on which forces are acting is balanced, the bar has no tendency to turn around any point. Instead, then, of choosing always the middle point, we might take any other convenient point.

If the algebraic sum of all the turning-tendencies with reference to this point is 0, there is no turning of the bar, or the bar remains in balance.

It must be noted, however, that once a certain point is selected from which to measure lever arms, this point must be retained throughout the solution.

To illustrate: In problem 5, above, let all lever arms be measured from the point F; i. e., choose F as the turning-point. Equation (2) then becomes:

$$(+f)(0)+(-2{,}400)(+\tfrac{1}{2})+(+x)(+6)=0;$$

whence $\qquad 6x-1{,}200=0,$

or, $\qquad 6x=1{,}200$, and finally $x=200$.

The first equation thus becomes unnecessary.

It is important to note that by a proper choice of the turning-point—say a point where some force acts which we do not care to know—problems may often be simplified.

6. Three boys desire to carry a 12 ft. log, weighing 240 lb. Two of the boys lift at the ends of a hand-spike placed cross-

wise underneath the log, and the third boy carries the rear end of the log. Where must the hand-spike be placed that all may lift equally.

A glance at the diagram shows that we have the case of a body (a log) balanced under the effect of three forces. De-

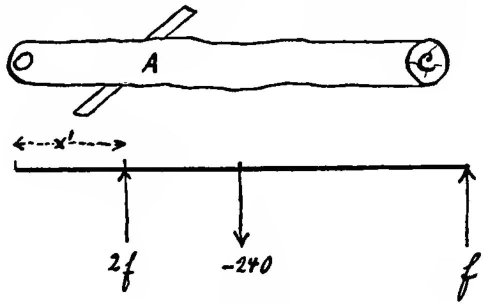

FIG. 43

note the force with which each boy lifts by f, and the distance from the end, O, of the log to the point, A, where the spike is placed, by x.

The three forces are $+2f$ at A, -240 at B, and $+f$ at C.

The problem will now be solved in two ways.

First, the point O, will be taken as the turning-point. By the conditions of the problem, the lever-arms will then be $+x$, $+6$, and $+12$.

The equations from Laws I and II of forces are:

$$(1) \qquad 2f-240+f=0,$$
$$(2) \quad (+2f)(+x)+(-240)(+6)+(+f)(+12)=0.$$

Complete the solution.

Second, the middle point B, will be taken as the turning-point. Let y denote the distance from the middle point, B, of the log to the point, A, where the spike is placed (draw the figure). The lever-arms are then $-y$, 0, and $+6$. The

first equation remains the same as above, but the second becomes:

$$(2)\quad (+2f)(-y)+(-240)(0)+(+f)(+6)=0,$$

which reduces at once to $\qquad -2fy+0+6f=0,$

or $\qquad -160y+480=0,$

$$\therefore \qquad y=3.$$

7. Solve problem 6 using the end, C, as the turning-point.

8. Solve problem 6 using A as the turning point.

NOTE.—Observe that the distance AB is $6-x$. This is the lever-arm of the weight of the log with reference to the turning-point A.

9. Suppose the length of the log to be l ft. and its weight 240 lb.; find where the spike should be placed that each boy may carry $\frac{1}{3}$ of the weight of the log.

10. Suppose the length of the log to be l ft. and its weight w lb. Find where the spike should be placed.

11. If the log be l ft. long and the weight w lb., where must the spike be placed if 4 boys, two at each end of the spike, are to lift at A, and a fifth at C, all to lift equally?

12. A steel beam 24 ft. long, and weighing 120 lb. per yd. is being moved by placing an axle borne by a pair of wheels under it, as shown at A, the end at B being carried. How far must the axle be placed from the end, O, and what will be the weight, W, on the axle, that the weight at B may be 200 pounds?

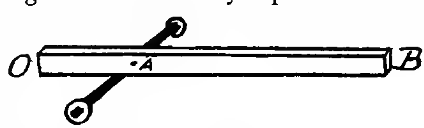

FIG. 44

Answer: $x=8\frac{16}{19}$ ft.

13. A steel rail, 30 ft. long, was supported at a point 14 ft. from one end. A lifting force of 45 lb. at the other end of the rail held the rail in balance. What was the weight of the rail

per yd. of length? Find the pressure on the supporting-point.

(Draw the figure.)

Answer: 72 lb.; 675 lb.

14. If the supporting-point of the rail (problem 12) had been $14\frac{1}{2}$ ft. from one end, what force at the other end would have balanced the rail? Find for this case the pressure on the supporting-point.

Answer: $23\frac{7}{31}$ lb.; $696\frac{24}{31}$ lb.

15. How may a railroad rail weighing more than a ton be weighed with a pair of balances running only to 60 pounds?

16. If the stone, S, presses down at the point, P, with a force of 4,250 lb., and the crow-bar, PA, 6 ft. long, is supported at a point, F, 4 in. from the end, P, what will be the pressure at F and the downward force at A when the bar is balanced?

FIG. 45

17. With other conditions as in 16, what would be the pressure at F and the force at A, if the distance from P to F were 3 inches?

18. A wheelbarrow is loaded with 45 bricks, averaging 6 lb. apiece. What lifting force will be needed at A to raise the load if the bar OA is $4\frac{1}{2}$ ft. and the distance from the center of the wheel, O, to the point, B, where the vertical line through the center of the load crosses OA, is 2 feet?

FIG. 46

19. With the same load and length of bar, OA, as in problem 18, how far is it from O to the crossing-point, B, of the vertical center line of the load, if 100 lb. at A just raises it?

20 A suction pump is a device for raising water from wells. The handle, OB, works against a pin at A, so that when the

hand pushes downward at B, the point O rises, and by the aid of a piston on the lower end, C, raises a mass of water. If OA = 2 in. and OB = 3 ft. what load at O will be raised by a force of 20 lb. pushing downward at B, and what will be the pressure on the pin at A?

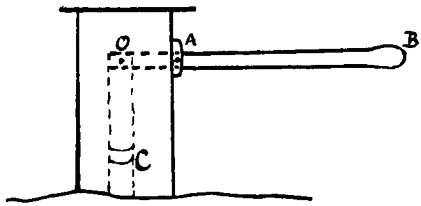

FIG. 47

21. With conditions as in 20, what force at O and pressure at A will be exerted by a downward force of 68 lb. at B?

22. A dry goods box weighing 360 lb. is being moved along the floor by the aid of a roller. If the box is 6 ft. long what

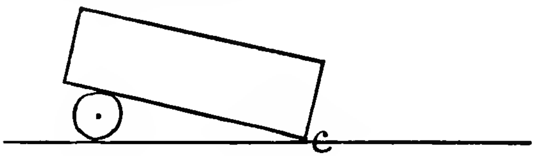

FIG. 48

force at C is needed just to raise the rear end, when the roller is 1 ft. from the front end? 2 ft. from the front end? 5 ft.? 5 in.? 6 ft.? 6 in.? What will be the weight on the roller in each case?

23. A foot-bridge 15′ long between supports (15′ span)

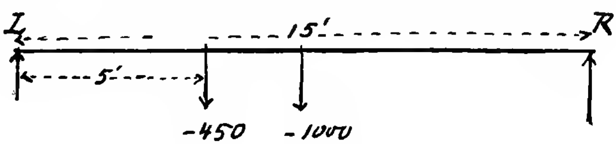

FIG. 49

rests on timbers at L and R. The bridge weighs 1,000 pounds. Two men, whose combined weight is 450 lb., stand just over

A, 5′ from the end L. Find the pressure on the supports at L and R.

The bridge may be regarded as a bar held in balance by 4 forces: l and r, called the reactions of the supports, push upward at the ends; the combined weight of the two men pushes downward at the point 5′ from the left end; and the weight of the bridge itself pulls downward at the middle of the span.

In the next two sections we shall find r and l first by the method of solving equations containing a single unknown and then, by the method of two unknowns.

§ 36. Solution by One Unknown Number

First, choose the turning-point at L and write down the equation of leverages, thus:

(1) $(+l)(0)+(-450)(+5)+(-1{,}000)(+7\frac{1}{2})+(+r)(+15)=0.$

Simplifying this equation:

$$-2{,}250-7{,}500+15r=0.$$

Adding 9,750 to both sides we have

$$(2)\quad 15r=9{,}750, \text{ whence } r=650.$$

Therefore the upward pressure of the right support against the bridge is 650 pounds.

Now choose the turning-point at r and write the equation of leverages, thus:

(3) $(+l)(-15)+(-450)(-10)+(-1{,}000)(-7\frac{1}{2})+(+r)(0)=0.$

Simplifying $\quad -15l+4{,}500+7{,}500=0.$

Adding $15l$ to both sides and then reversing the sides of the equation we have:

$$(4)\quad 15l=12{,}000\,; \quad \text{whence } l=800.$$

Therefore the upward pressure of the left support against the bridge is 800 pounds.

§ 37. Solution by Two Unknown Numbers

To solve the problem of § 36 by the method of two unknown numbers, both of the laws of force will be needed, the turning-point being now taken at the middle point, C, of the span.

The equations furnished by the two laws are:

(1) $+L-450-1{,}000+R=0$ (state the law),

(2) $(+L)(-7\frac{1}{2})+(-450)(-2\frac{1}{2})+(-1{,}000)(0)$
$+(+R)(+7\frac{1}{2})=0$ (state the law).

The latter equation becomes

$$-7\tfrac{1}{2}L+1{,}125+7\tfrac{1}{2}R=0.$$

Multiplying this equation through by 2 and subtracting 2,250 from both sides, we find

$$15R-15L=-2{,}250.$$

Dividing through by 15, we get $R-L=-150$. Adding 1,450 to both sides of the equation (1) gives

$$R+L=1{,}450.$$

We now have
$$\begin{cases} R+L=+1{,}450, \\ R-L=-150. \end{cases}$$

Adding these two equations, member to member (the sides of an equation are often called its members), we have:

$$2R=1{,}300, \quad \text{or } R=650.$$

Subtracting the same two equations, member from member, we find:

$$2L=1{,}600, \quad \text{or } L=800.$$

Hence, the left support presses upward against the bridge with a force of 800 lb., and the right with 650 lb., as we found before.

It is well to notice that by a proper selection of the turning point of leverages, the problem may be solved by either one of the above methods.

1. If, in the last example just solved, the 450 lb. weight had been 6 ft. from the left support, what would have been the pressures on the supports?

2. A bridge 20 ft. long weighs 2,400 lb., and supports two loads; one of 600 lb., 4 ft. from the left end, and the other 800 lb., 15 ft. from the left. What are the loads borne by the supports?

3. If, with the bridge of problem 23, § 35, four loads of 450 lb. each are placed, one 2′ from the left support, the second 6′, the third 9′, and the fourth 16′ from the left end, what are the upward forces exerted against the ends of the bridge by the supports?

4. A wagon box, EFGO, 10′ long and loaded with 40 bu. of wheat weighing 60 lb. per bu., extends $1\frac{1}{2}'$ in front of the front axle, A, and 2′ behind the rear axle, B. What is the load on each axle?

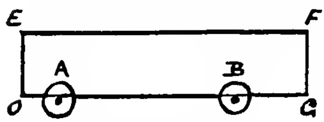

FIG. 50

5. If the box (problem 4) is filled with coal, it will hold two tons. What load on each axle would this weight produce?

6. A box 12′ long of a three-horse coal wagon is loaded with 6 tons of coal. If the box extends 2′ in front of the front axle and 4′ back of the rear axle, what are the weights on the front and rear axles?

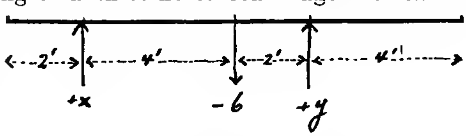

FIG. 51

7. A lumber wagon is coupled out to a distance of 9′ between the axles and loaded with a pile of lumber $3\frac{1}{2}' \times 4' \times 18'$. The load extends 3′ in front of the front axle and the material averages 48 lb. per

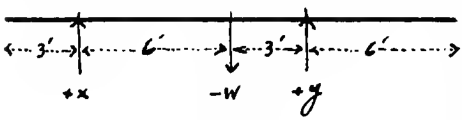

FIG. 52

cu. ft. What are the pressures on the axles due to this load? (Draw the figure.)

8. The water tank of a park sprinkler is a circular cylinder 51 in. in diameter and 5 ft. high. A driver, weighing 180 lb., sits at A, directly above the front axle. The vertical center line through C passes between the wheels 3 ft. in front of the rear axle, and 4 ft. behind the front axle. If water weighs $62\frac{1}{2}$ lb. per cu. ft., what is the weight on each axle, when the tank is level full and the driver is on the seat?

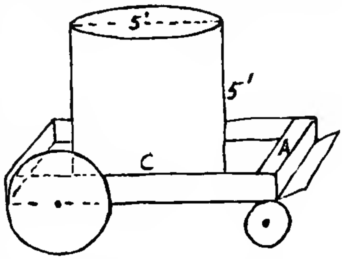

FIG. 53

9. A wagon standing on a culvert, AB, has on its front axle a load of 2,500 lb., and on its rear axle, 3,000 lb. The front wheels are 4′ and the rear wheels 10′ from the left end of the culvert. If the culvert is 20′ long, what are the pressures at the supports?

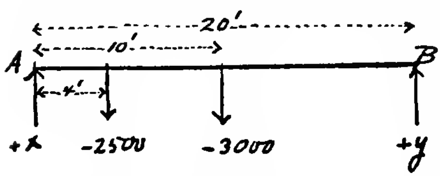

FIG. 54

10. What would these pressures be if the front wheels stood 9′ from the left support (wagon coupled to 6′ between the axles)?

CHAPTER XII

THE SIMPLE EQUATION

§ 38. The Axioms

A light bar, AB, is carefully balanced at C. Equal weights, w, w', are then hung at equal distances on opposite sides of C, as shown in Fig. 55. The beam remains balanced under these weights.

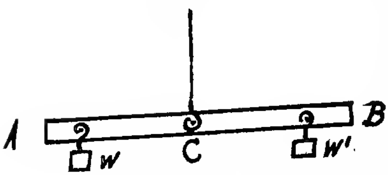

FIG. 55

1. Two 4-oz. weights are now hung to w. How many ounces must be hung to w' for balance?

2. If another 4-oz. weight is hung to w, how many ounces must be hung to w' for balance?

3. If x oz. are hung to w, how many ounces must be hung to w' for balance?

Show that these problems illustrate the truth of the following:

ADDITION AXIOM.—*If the same number, or equal numbers, be added to equal numbers, the sums are equal.*

4. Hang six 4-oz. weights on w. How many 2-oz. weights must be hung on w' for balance?

5. If a 4-oz. weight is now removed from the left side how many 2-oz. weights must be taken off the right side to restore the balance?

6. If two more 4-oz. weights are taken from the left, how many 2-oz. weights must be taken from the right for balance?

7. If the bar is balanced at first by a pair of equal weights of any number of pounds and any number of pounds is taken

from the left, how many pounds must be taken from the right for balance?

8. If the bar is balanced under x lb. on each side of C, and y lb. is taken from the left, how many pounds must be taken from the right for balance?

What has been suggested in the last problem may be written thus:

$$
\begin{array}{rl}
(1) & x=x \\
(2) & \underline{y=y} \\
(3) & x-y=x-y.
\end{array}
$$

Show that the last five problems illustrate the following:

SUBTRACTION AXIOM.—*If the same number, or equal numbers, be subtracted from equal numbers, the differences are equal.*

9. Suppose the bar of Fig. 55 is first balanced by a 2-oz. weight on each end. If another 2-oz. weight is added on the left, how many 2-oz. weights on the right will balance the bar?

10. Suppose the bar is balanced under any pair of equal weights. If the weight on the left is doubled what change must be made on the right for balance? What change must be made on the right if the weight on the left is trebled? quadrupled? What change must be made on the right if the weight on the left is multiplied by 6? by 12? by 25? by a?

Show how these questions illustrate the truth of the following:

MULTIPLICATION AXIOM.—*If equal numbers be multiplied by the same, or equal, numbers, the products are equal.*

In problems 11, 12, and 13, suppose the bar of Fig. 55 is balanced by 12 2-lb. weights on each end.

11. If half as many weights are placed on the left, what part of the number of weights on the right must remain for balance?

12. If $\frac{1}{3}$ of the weights are on the right, what part must remain on the left for balance?

13. What part on the right must remain for balance, if $\frac{1}{4}$ of those on the left remain? if $\frac{1}{6}$? if $\frac{1}{12}$?

These exercises make clear the following axiom.

DIVISION AXIOM.—*If equal numbers be divided by the same number, or by equal numbers, the quotients are equal.*

Exercises on the axioms.

14. The bar is balanced under $3x$ lb. on one side and 9 lb. on the other. Six pounds additional are now hung to each side and the bar still balances. This is expressed in algebraic language thus:

$$\begin{array}{ll} (1)\ 3x=9 & \\ (2)\ 6\ \ =6 & \text{(Any number may be written equal to itself.)} \\ \hline 3x+6=15\,. & \end{array}$$

15. If $x=8$ and $a=6$ what third relation may be written from the addition axiom? What from the subtraction axiom? The multiplication axiom? The division axiom?

16. If $4x-5=7$ and we write $5=5$, what third relation may be written from the addition axiom? From the multiplication axiom?

17. If $x-11=10$, show how to find the value of x by the addition axiom.

18. If $x+5=9$, show how to find the value of x by the subtraction axiom.

19. If $4x=12$, show how to find the value of x by the division axiom.

20. If $\frac{1}{2}x=6$, show by the multiplication axiom how to find what x equals.

21. Give the reason for the truth of the third statement in each of the following cases:

If, $a=9$ and $b=3$, then $a+b=12$;
$m=13$ and $n=2$, then $mn=26$;
$a=x$ and $b=y$, then $a+b=x+y$;
$c=16$ and $d=4$, then $c-d=12$;
$m=28$ and $n=7$, then $\frac{m}{n}=4$;
$\frac{c}{12}=5$ and $\frac{d}{12}=3$, then $\frac{c+d}{12}=8$.

22. Give the reasons for the following:

If $a=7$,	$5a=35$;	If $2y=6$,	$y=3$;
$\frac{f}{16}=3$,	$f=48$;	$x-5=3$,	$x=8$;
$mn=6n$,	$m=6$;	$\frac{r}{10}=7$,	$r=70$.

§ 39. Solution of Equations by Axioms

1. If $5x+9=3x+17$, find the value of x.

SOLUTION.—

$$\begin{array}{rll} & 5x+9=3x+17 & \\ & 3x \quad\;\, =3x & \text{(Any number may be written equal to itself.)} \\ \hline \therefore & 2x+9=17 & \text{Why?} \\ & 9=9 & \text{Why?} \\ \hline \therefore & 2x=8 & \text{Why?} \\ & x=4 & \text{Why?} \end{array}$$

and the value of x is 4.

Check: $5\cdot 4+9=3\cdot 4+17=29$.

2. $5x-17+3x-5=6x-7-8x+115$. Find x.

SOLUTION.—Collecting like terms on each side,

$$\begin{array}{rll} & 8x-22=-2x+108 & \\ & 22=22 & \text{Why?} \\ \hline & 8x=-2x+130 & \text{Why?} \\ & 2x=2x & \text{Why?} \\ \hline & 10x=130 & \text{Why?} \\ \text{and} & x=13 & \text{Why?} \end{array}$$

Check this result.

Solve the following equations for the values of x and check all results:

3. $3x+15=x+25$.
4. $2x-3=3x-7$.
5. $7x-39-10x+15=100-33x+26$.
6. $118-65x-123=15x+35-120x$.
7. $\frac{3x}{4}+5=91-10x$.
8. $8+2x+\frac{x}{4}=1\frac{3}{4}+\frac{2x}{3}$.
9. $3(x-2)+15=5x-3$.
10. $3x-2(x+5)=6x-20$.
11. $5(x-1)=3(x+1)$.
12. $3x+14-5(x-3)=4(x+3)$.
13. $2(x-1)+3(x-2)=4(x-5)$.
14. $2(x+1)+3(x+2)+4(x+3)=110$.
15. $5(x-2)+6(x-1)-4(x-5)=60$.
16. $2(x-3)+3(2x+1)-2(3x-1)=47$.
17. $4(2x+9)+3(x-9)-5(2x-7)=13$.
18. $x(x-1)-x(x-2)=4(x-3)$.

CHAPTER XIII

THE GRAPH

§ 40. Locating Points by Means of Numbers

Choose some intersection on a piece of squared paper to represent the position of the central point of a city and let one small square represent one city-block. From this fixed reference point, or origin, let the position of an object m blocks east (to the right on the paper) be denoted by $+m$E, and m blocks west (to the left) by $-m$E. Then if an object is situated at +4E, it is somewhere on the street 4 blocks east of the central point or 4 blocks east of the north and south street through the central point. We will call the north and south street through the origin the N-S axis, and the east-west street through the origin the E-W axis. Similarly, an object at −2E is somewhere on the street 2 blocks west of the origin, or of the N-S axis. The exact north and south positions of the object are, however, not given in either case.

1. Let the position of an object k blocks north (upward on the paper) be denoted by $+k$N; and k blocks south (downward on the paper), by $-k$N. Then if an object is situated at −7N, it is somewhere on the street 7 blocks south of the origin, or the E-W axis, and at +5N, it is somewhere on the street 5 blocks north of the origin, or the E-W axis. In this case the exact east and west positions of the objects are unknown. Point out the position of an object at +3E and at the same time at −7N (See Fig. 56, p. 84). How many points are so situated?

2. Point out the positions of the following places:

+11E, −7N;	+2E, +9N;
$+3\frac{1}{2}$E, +5N;	+2E, −9N;
−5E, −6N;	−2E, +9N;
−4E, −3N;	−2E, −9N.

3. If it is agreed that the E-distance shall always be written first, and the N-distance second, the position of a point may be

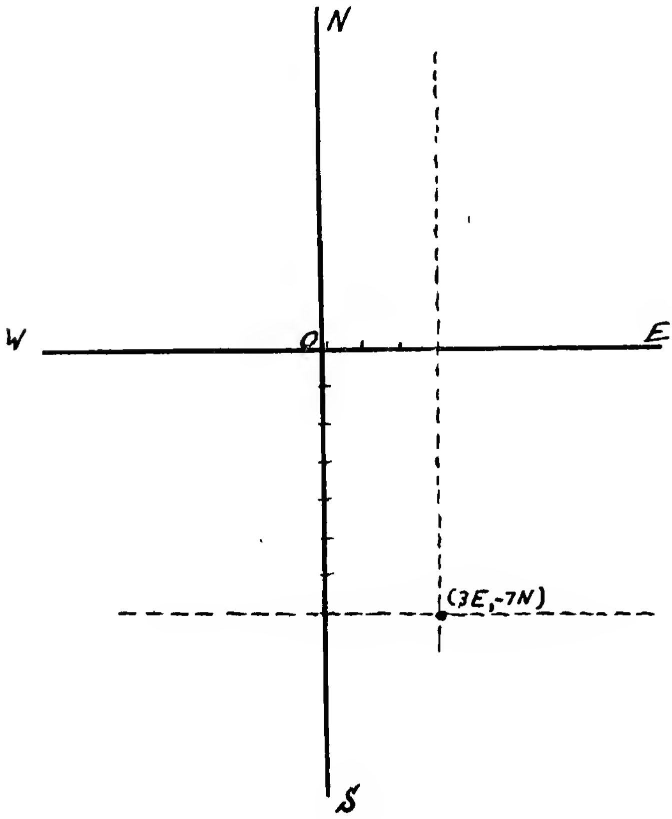

FIG. 56

represented by two numbers with their signs; i. e., $+11$E, -7N may be written simply $+11$, -7.

Locate the following points:

$+7, -5;$	$-8, -3;$	$0, -4;$
$-6, -4;$	$-11, +11;$	$0, +3;$
$-13, +9;$	$+6, +3;$	$-3, 0;$
$+4, +6;$	$+8, -7;$	$0, 0.$

4. Locate the following points:

$+3, +1;$	$+12, +4;$	$-3, -1;$	$+6, +2;$
$+9, +3;$	$+18, +6;$	$0, 0;$	$-6, -2.$

5. How do the points in problem 4 lie with respect to each other? What is the relation between the E-distance and the N-distance for all the points. Draw the line passing through the points. Take some other points on the line and measure their co-ordinates (i. e., their E-distance and their N-distance). Do their values have the same ratio to each other as the co-ordinates of the points in problem 4?

§ 41. Picturing or Plotting Laws Connecting Numbers

1. Find ten points for which $N=5E$ and locate, or plot, their positions on the squared paper. How do they lie with respect to each other? Make a table of the co-ordinates of the points as follows:

E	N
-2	-10
-1	-5
0	0
$+1$	$+5$

2. In the relation $N=2E+5$, let E take all the integral values (whole numbers) from -7 to $+7$. Make a table of the points and plot them. Do they all lie in a single straight line? What is the least number of points from which the position of such a straight line may be fixed?

NOTE.—A line determined as in problem 2 from an equation is said to *represent* the equation.

3. Plot, from three points (two to fix the position of the line, and the third as a check) the straight lines which represent the following equations:

(1) $N=5-2E$;
(2) $N=\frac{3}{5}E$;
(3) $N=\frac{2}{3}+\frac{E}{5}$;
(4) $N=\frac{3-4E}{2}$;
(5) $2N+5E=4$;
(6) $2N-3E=5$.

4. In each part of problem 3 measure the E-distance from the origin to the point where the graph of the equation cuts the E-W axis. What is the value of N for this point?

It is evident that for $N=0$ in, say, problem 3,

$$(1) \qquad 5-2E=0,$$
$$\text{and} \qquad E=2\tfrac{1}{2}.$$

Thus from the graph of the equation $N=5-2E$, we may obtain the solution of the equation $5-2E=0$.

For example, to solve the equation

$$\frac{3E}{2}+11=4E+19,$$

transpose, or change all terms to the right side of the equation by means of the axioms and simplify, whence

$$0=5E+16.$$

Put the expression on the right equal to N, thus:

$$N=5E+16.$$

Plot the graph of this equation and measure the distance from the origin to the point where the graph cuts the E-W axis. This gives the value of E for which $N=0$, or for which

$$5E+16=0,$$

i. e., we get the solution of the equation

$$\frac{3E}{2}+11=4E+19.$$

This method of solution is called the *graphical method*.

5. Solve the following by the graphical method:

(1) $\frac{E}{5}+2E-9=\frac{11}{3}+5E$;

(2) $5E+11+9E-2=6E-5$;

(3) $4(3-2E)=7(11E-5)$;

(4) $5(2E-4)+8(E-4)=17$;

(5) $3(E-3)-5(2E-5)=9(E+2)-4(2E+3)$.

6. Suppose the E-W axis is called the x-axis and the plus and minus E-distances are called plus and minus x-distances, and the N-S axis is called the y-axis and the plus and minus N-distances are called plus and minus y-distances. Assuming the first number in each pair to be the x-distance, plot the following points:

$-5, +4$;	$-7, +15$;	$-9, -8$;	$0, 0$;
$-3, +11$;	$+5, +12$;	$+8, -1$;	$0, -4$.

7. Tabulate from the following equations the values of y for all integral values of x from -5 to $+5$; plot the graph, and measure the value x for which $y=0$:

(1) $y=\frac{3x}{7}-5$;

(2) $y=3(2x-5)$;

(3) $y=5(7-4x)-18$;

(4) $5y=6(2-3x)-5(1-x)$.

8. Solve graphically the following equations, and check each result by solving algebraically by means of the axioms:

(1) $2x+3=16-(2x-3)$;

(2) $8(10-x)=5(x+3)$;

(3) $2x-3(2x-3)=1-4(x-2)$;

(4) $\frac{3x}{5}-\frac{x}{6}=\frac{26}{15}$.

CHAPTER XIV

EQUATIONS CONTAINING FRACTIONS

§ 42. Freeing an Equation of Denominators

1. To find the value of x in an equation of the form

$$\frac{3x}{7}=12,$$

it is necessary to clear the equation of fractions by multiplying both sides of the equation by the denominator 7 (multiplication axiom); then

$$3x=84,$$

and $$x=28.$$

Find x in $\frac{8x}{9}=16$.

2. If the equation involves several fractions it may be simplified by multiplying both sides of the equation by a number which will contain, as factors, all of the denominators; e. g., to solve

$$\frac{x}{2}-\frac{2x}{5}+\frac{3x}{7}=\frac{37}{35},$$

both sides of the equation may be multiplied by 70, giving

$$35x-28x+30x=74,$$

whence $$37x=74,$$

and $$x=2.$$

Find x in $\frac{x}{3}+\frac{3x}{4}-\frac{9x}{8}=-1$.

3. In simplifying an equation involving several fractions it is best to multiply both sides of the equation by the least number which contains, as factors, all of the denominators; i. e., by the least common multiple of the denominators, called

the least common denominator. The least common denominator is found, as in arithmetic, by taking the product of the different prime factors of all the denominators, each factor being used the greatest number of times it occurs, as a factor, of any one denominator; e. g.:

$$(1)\quad \frac{x}{24}-\frac{5x}{42}+\frac{11x}{56}=\tfrac{17}{36},$$

$$24=2\times2\times2\times3,$$
$$42=2\times3\times7,$$
$$56=2\times2\times2\times7.$$

Two occurs, at most, three times; three, at most, twice; and seven, at most once, in the factors of any one denominator, therefore $2^3\times3^2\times7$ is the *least common multiple* of the numbers 24, 56, and 36, or the *least common denominator* of the given *denominators*.

SOLUTION.—Multiply both sides of the equation (1) by $2^3\times3^2\times7$, and

$$21x-60x+99x=14\times17,$$

complete the solution.

It is advisable in clearing equations of fractions to avoid multiplying out results until absolutely necessary.

QUERY.—Could an equation be freed of its denominuators by multiplying both sides by any other number or numbers, than the least common multiple of the denominators?

§ 43. Problems in Equations Involving Fractions

1. Simplify the following equations and find the value of x:

$$(1)\quad \frac{2x}{3}-\tfrac{5}{6}=\frac{x}{4};$$

$$(2)\quad \frac{x-8}{5}=\tfrac{7}{13};$$

$$(3)\quad \frac{x-8}{7}+\frac{x+3}{3}+\tfrac{5}{21}=0;$$

(4) $\dfrac{x-3}{7}+\dfrac{x+5}{3}-\dfrac{x+2}{6}=4$;

(5) $\dfrac{2x-5}{5}+\dfrac{x-3}{2x-15}=\dfrac{4x-3}{10}-1\tfrac{1}{10}$;

(6) $\dfrac{4(x+3)}{9}=\dfrac{8x+37}{18}-\dfrac{7x-29}{5x-12}$;

(7) $\dfrac{3x+5}{2x+3}-\dfrac{6x-1}{4x-1}=0$;

(8) $\dfrac{3+x}{5x-3}-\dfrac{x+1}{5x+2}=0$;

(9) $\dfrac{1}{1-x}+\dfrac{6}{x}=\dfrac{6+x}{1-x}$.

2. Solve the following literal equations:

(1) $\dfrac{x-b}{a}=\dfrac{x-a}{b}$.

Solution.—Multiply both sides by ab, the least common multiple of the denominators, and obtain

$$bx-b^2=ax-a^2.$$

Transpose the terms containing x to the left side and all other terms to the right side of the equation:

$$bx-ax=b^2-a^2,$$

or, $\quad x(b-a)=b^2-a^2 \quad \left(\text{since } (b-a)x=bx-ax\right)$,

dividing by $(b-a)$, $x=b+a$ $\left(\text{since } (b+a)(b-a)=b^2-a^2\right)$.

(2) $\dfrac{x}{a}+\dfrac{x}{2a}=3$;

(3) $\dfrac{x}{a}-\dfrac{2x}{3a}=\tfrac{1}{2}$;

(4) $\dfrac{mx}{n}+\dfrac{nx}{m}=m^2+n^2$;

(5) $\dfrac{1}{m}+\dfrac{1}{x}=\dfrac{1}{n}-\dfrac{1}{x}$;

(6) $cx+a+\dfrac{x}{c}=\dfrac{x}{a}$;

(7) $\dfrac{x+m}{x-n}=\tfrac{3}{4}$;

(8) $\dfrac{4}{m+x}=\dfrac{3}{m-x}$.

(9) Solve (1) for a; for b.

(10) Solve (2) for a.

(11) Solve (4) for m; for n.

(12) Solve (8) for m.

§ 44. Problems Leading to Equations Containing Fractional and Literal Numbers

1. Bell metal is by weight five parts tin and 16 parts copper. How many pounds of tin and of copper are there in a bell which weighs 4,200 lb.?

Solution.—*First method:* Let x be the number of lb. of tin, then $\frac{16x}{5}$ is the number of pounds of copper.

Then, $$x+\frac{16x}{5}=4{,}200,$$
whence, $$x=1{,}000,$$
and $$\frac{16x}{5}=3{,}200.$$

Therefore there are 1,000 lb. of tin and 3,200 lb. of copper.

Second method (without fractions): Let $5x$ be the number of lb. of tin, then $16x$ is the number of lb. of copper;

so that $$5x+16x=4{,}200,$$
whence $$x=200.$$
Then, $$5x=1{,}000,$$
and $$16x=3{,}200.$$

Therefore there are 1,000 lb. of tin and 3,200 lb. of copper.

2. Gunpowder contains by weight 6 parts saltpeter, 1 part sulphur, and 1 part charcoal. How many pounds of saltpeter, of sulphur, and of charcoal are there in 120 lb. of gunpowder?

3. A certain compound contains by weight 5 parts of carbon to every 3 parts of iron, and 7 parts of iron to every 2 parts of copper. In 124 lb. of the compound how many pounds are there of carbon, of iron, and of copper?

4. In an alloy of silver and copper weighing 90 oz. there are 6 oz. of copper; find how much silver must be added so that 10 oz. of the new alloy shall contain $\frac{2}{5}$ of an ounce of copper.

5. If 80 lb. of sea-water contain 4 lb. of salt, how much fresh water must be added to make a new solution of which 45 lb. contain $\frac{2}{3}$ lb. of salt?

6. One bin contains a mixture of 14 bu. of oats and 16 bu. of wheat. Another contains 20 bu. of oats and 12 bu. of wheat. How many bushels must be taken from each bin to make a mixture of 5 bu. of oats and 5 bu. of wheat?

7. If 1 lb. of lead loses $\frac{5}{57}$ of a lb., and 1 lb. of iron loses $\frac{2}{15}$ of a pound, when weighed in water, how many pounds of lead and of iron are there in a mass of lead and iron that weighs 159 lb. in air and 143 lb. in water?

8. If a mass of gold weighs 97 oz. in air and 92 oz. in water, and a mass of silver weighs 21 oz. in air and 19 oz. in water, how many ounces of gold and of silver are there in a mass of gold and silver that weighs 320 oz. in air and 298 oz. in water?

9. Into what two sums can $1,000 be divided so that the income of one at 6 per cent. shall be equal to the income of the other at 4 per cent.?

10. What percentage of evaporation must take place from a 6 per cent. solution of salt and water (salt water of which 6 per cent. by weight is salt) to make the remaining portion of the mixture an 8 per cent. solution? 12 per cent. solution?

11. A physician having a 6 per cent. solution of a certain kind of medicine wishes to dilute it to a $3\frac{1}{2}$ per cent. solution. What percentage of water must be added to the present mixture?

12. An express train whose rate is 40 mi. per hour starts 1 hr. and 4 min. after a freight train, and overtakes it in 1 hr. and 36 minutes. How many miles does the freight train run per hour?

SUGGESTIONS.—Let x be the rate of the freight train in miles per hour. Both trains travel the same distance, therefore

$$2\tfrac{2}{3}\times x=40\times 1\tfrac{3}{5},$$

$$\frac{8x}{3}=64,$$

$$x=24.$$

Hence the freight train runs 24 miles per hour.

13. Two trains start at the same time from S, one going east at the rate of 35 mi. per hr. and the other going west at a rate $\frac{1}{7}$ faster. How long after starting will they be exactly 100 mi. apart?

14. A man walks beside a railway at the rate of 4 mi. per hr. If a train 208 yd. long, traveling 30 mi. per hr., overtakes him, how long will it take the train to pass the man?

15. A railroad train moves at a uniform rate. If the rate were 6 mi. per hr. faster, the distance it goes in 8 hr. would be 50 mi. greater than the distance it would go in 11 hr. at a rate 7 mi. per hr. less than the actual rate. Find the actual rate of the train.

16. Two trains go from P to Q on different routes, one of which is 15 miles longer than the other. A train on the shorter route takes 6 hr. and a train on the longer, running 10 miles less per hr., takes $8\frac{1}{2}$ hr. Find the length of each route.

17. The distance from A to B is 100 miles. A train leaving A at a certain rate meets with an accident 20 miles from B, reducing its speed one-half, and causing it to reach B an hour late. What was its rate per hour before the accident?

18. A man rows down stream at the rate of 6 mi. per hr. and returns at the rate of 3 mi. per hr. How far down stream can he go and return within 9 hr.?

19. A boatman moves 5 mi. in $\frac{3}{4}$ of an hour, rowing with the tide; to return over the same route it takes him $1\frac{1}{2}$ hr., rowing against a tide one-half as strong. What is the velocity of the stronger tide?

20. A boatman, rowing with the tide, moves a mi. in b hours. Returning, it takes him c hr. to cover the same distance, rowing against a tide m times as strong as the first. What is the velocity of the stronger tide?

21. The distance from A to B is 32 miles. A man sets out from A and reaches B 12 minutes after a second

man, who left A when the first man had proceeded 11 miles upon his journey. If the second man can make the journey in 4 hours, at what distance from B did he pass the first man?

22. A and B run a race. At the end of 5 minutes when A has run 900 yd. and is ahead of B by 75 yd., he falls and for the rest of the race makes 20 yd. less per minute than at first. He comes in one-half minute behind B. What was B's time?

23. A bicyclist traveling a mi. per hr. is followed, after a start of m miles by a second bicyclist, traveling b mi. per hr., $(b>a)$. At these rates, in how many hours after the second starts will he overtake the first?

24. At what time between three and four o'clock are the hands of the clock together?

SUGGESTION.—Let x represent the number of minute spaces over which the minute hand passes from three o'clock on until it first overtakes the hour hand; then show that $\frac{x}{12}+15$ represents the same number of spaces; whence the equation is:

$$x=\frac{x}{12}+15,$$

$$x=16\tfrac{4}{11}:$$

Hence the hands are together at $16\frac{4}{11}$ minutes past 3 o'clock.

25. At what time between two and three o'clock are the hands of the clock together?

26. At what time between 3 and 4 o'clock are the hands of the clock at right angles (two results)?

27. At what time between 7 and 8 o'clock are the hands of the clock pointing in opposite directions?

28. At what time between 5 and 6 o'clock are the hands of the clock one minute space apart? (Two results.)

29. If the second hand of a watch has made $\frac{5}{11}$ of a revoultion and the minute and hour hands of the watch are together, what time is it?

30. The planet Venus passes around the sun 13 times to the earth's 8. How many months is it from the time when Venus is between the earth and the sun to the next time when it is in the same relative position?

31. Seen from the earth, the moon completes the circuit of the heavens in 27 da., 7 hr., 43 min., 4.68 sec., and the sun in 365 da., 5 hr., 48 min., 47.8 sec., in the same direction. Required the time to 0.0001 da. from one full moon to the next, the motion supposed to be uniform.

32. The difference between the squares of two consecutive numbers is 19. Find the numbers.

33. A box of oranges was bought at the rate of 15 cents a dozen. Five dozen were given away and the remainder sold at the rate of 2 for 5 cents. If this gave a profit of 30 cents on the box, how many were there in the box?

34. A company of men was drawn up in a hollow square 4 deep. Afterward it was separated into two detachments. One was a hollow square 3 deep and with 34 more men in front than formerly, and the other contained 80 men. How many men in the company?

35. A man engaged to work 20 days on these conditions: for each day he worked he was to receive 2 dollars, and for each day he was idle he was to forfeit 1 dollar. At the end of 20 days he received 34 dollars. How many days was he idle?

36. A man engaged to work a days on these conditions: for each day he worked he was to receive b dollars; and for each day he was idle he was to forfeit c dollars. At the end of a days he received d dollars. How many days was he idle?

37. The circumference of the fore wheels of a carriage is 9 ft.; that of the hind wheels, 12 feet. What distance will the carriage have passed over when the fore wheels have made 5 more revolutions than the hind wheels?

38. The circumference of the fore wheels of a carriage is a ft.; that of the hind wheels, b ft. What distance will the carriage have passed over when the fore wheels have made n more revolutions than the hind wheels?

39. A man spends one ath of his income for food, one bth for rent, one cth for clothing, one dth for furniture and he saves e dollars. How much is his income?

CHAPTER XV

THE FUNDAMENTAL PROCESSES APPLIED TO INTEGRAL ALGEBRAIC EXPRESSIONS

§ 45. Addition of Integral Algebraic Expressions

An algebraic expression whose parts are numbers connected by addition or subtraction, or by both, is called a *polynomial.* The parts of the polynomial, together with the signs preceding them, are called terms.

How many terms are in $\frac{a}{b}+5ax+\frac{7c}{m}$?

How many terms are in $-6+\frac{1}{2}x+7y\div 3x-4$?

If a term is composed of factors, any one of the factors, or the product of several factors, is called the coefficient of the product of the other remaining factors; e. g., in the term $3ax$

the coefficient of x is $3a$,
" " " ax is 3,
" " " $3x$ is a,
" " " 3 is ax.

What is the coefficient of b in $3abx$, $2abcx$, $4abxy$, $\frac{-6abxz}{9c}$? Of abx in the same expressions?

What is the coefficient of ax in the same expressions?

Show by counting of units that $2a+3a=5a$, $5a+2a+7a=14a$.

Unite $3x+5x$ into one term; also $7y+9y-3y$.

Give a single term equal to $xy+xy$, or $\frac{2}{3}x-\frac{1}{3}x$, or $5xy+8xy-2xy$, or $2(x-y)+4(x-y)-3(x-y)$.

Can $\frac{1}{3}x+\frac{2}{3}y$ be united into one term?

Under what condition is it possible to unite terms into a single term? Make a rule for adding numbers of the same denomination.

1. Express the following in as few terms as possible:

(1) $7a+(-5a)$;	(6) $20x+(-25x)$;
(2) $10a+(-7a)$;	(7) $-3b+(-4a)$;
(3) $16y+(-9y)$;	(8) $7m+(-16p)+(-3m)+(-12p)$;
(4) $-8a+17a$;	(9) $16x+21x+(-11x)$;
(5) $-9k+12k$;	(10) $-7y+(-8y)$.

2. To 6 times a given number add 12 times that number, then add to this sum -7 times the given number. What is the result?

3. The lengths of two lines are $5i$ and $3i$. What is their sum? Find the sum both algebraically and geometrically.

4. A man mixes $27g$ of saltpeter with $12g$ of sulphur and $12g$ of charcoal. How much gunpowder does he get?

5. The property of a man consists of $200x$ acres in fields, $80x$ acres in meadow, $60x$ acres of woodland, and $20x$ acres of garden. How many acres does he own in all?

6. Add the following numbers:

(1) $6n$, $7n$, $-3n$, $18n$, and $-11n$;
(2) $5a^2x$, $-3a^2x$, a^2x, and $-3a^2x$;
(3) $3mp^2$, $-8mp^2$, $5a^2x$, $-4mp^2$, $-3a^2x$, and $2ax^2$;
(4) $23a^2$, $5b^2$, $-8a^2b^2$, $-13b^2$, $24a^2b^2$, and $-19a^2$;
(5) a^2bx, $12a^3bx$, and $-9a^3x$;
(6) $-15(ax^2+3)$, $27(ax^2+3)$, and $-9\ (ax^2+3)$;
(7) $13(x-y)$, $-a(x-y)$, $\frac{5}{7}m(x-y)$;
(8) mxy and $+nxy$;
(9) ax^2, bx^2, $-cx^2$, $-dx^2$;
(10)
$$\begin{array}{l} 5a+\ \ 3b+9c \\ \underline{2a-10b-2c}\,; \end{array}$$
(11) $6a+15c-17b-8d$ and $-7a+12d+15b-12c$;
(12) $7a^2+23b^2-15c^2$ and $-21b^2-6a^2+12c^2$;
(13) $29xy-27y^2+24x^2$ and $-14xy-36xz+36y^2$;
(14) $-3a-7b+14c$ and $-11a+20b-34c$;
(15) $14k-11l+12m$, $-3k+12l-6m$, and $-12k+l-2m$;
(16) $-18a^2b^2+12a^4-8a^3b$ and $4\frac{1}{2}ab^3-a^2b^2+3\frac{3}{8}a^3b$.

§ 46. Subtraction of Integral Algebraic Expressions

If two numbers, as 2 and 5, are given, then 7 can be found from 2 and 5 by addition. On the other hand, if 7 and 5 are given, then 2 can be found as *that number which added to* 5 *gives* 7. If 2 and 7 are given, 5 can be found *as that number which added to* 2 *gives* 7.

DEFINITION.—To subtract the number b from the number a means to find that number which added to b produces a; i. e., to find the number of units which are to be counted from b to get to a.

To indicate that b is to be subtracted from a the *symbol* $(-)$ is written between b and a, as $a-b$, which is read "a minus b."

The process of finding a number which, added to a given number, produces a second given number is called *subtraction.*

1. According to the definition of subtraction find the values of the following differences:

(1) $9-5$;
(2) $6a-2a$;
(3) $8^\circ-6^\circ$;
(4) What number added to $-b$ gives a?

Find the following differences:

(5) $a-(-b)$; (8) $9a-(-3a)$;
(6) $16-(-3)$; (9) $10x-(-5x)$;
(7) $21-(-15)$; (10) $25-(-9)-(-6)$;
(11) $10y-(-3y)-(-4y)$;
(12) What number added to $+b$ gives $-a$?
(13) Find $-a-(+b)$; (16) $-18a-(+16a)$;
(14) Find $-5-(-3)$; (17) $-27y-(+36y)$;
(15) $-17b-(+15b)$; (18) $-6cb-(+3cb)$;
(19) What number added to $-b$ gives $-a$?
(20) Find $-a-(-b)$; (24) $-17s-(-4s)$;
(21) $-11b-(-7b)$; (25) $-8u-(-8u)$;
(22) $-7p-(-7p)$; (26) $-6cd-(-cd)$;
(23) $-12q-(-5q)$; (27) $-11zl-(-16zl)$.

Make a rule for the subtraction of positive or negative numbers.

2. According to this rule, subtract the lower number of each of the following pairs from the upper number:

13	-7	-75	$+9$	12	x	$4+u$
-5	-8	$+25$	$+21$	-5	-5	-2

$7a$	$16bx^2$	$-18m^2$	$-18x^2v^2$	$26v^3y^5$	$15cx^n$
$4a$	$-3bx^2$	$5m^2$	$-5x^2v^2$	$-7v^3y^5$	$3cx^n$

$6m^2px$	$5(a-2b^3)$	$15m^2(x-y)$	$-2x(1+5a^2y)$
$-4m^2px$	$-11(a-2b^3)$	$-23m^2(x-y)$	$-14x(1+5a^2y)$

3. Add p^3+3p^2+4p-6, $-p^2-2p+1$, p^2-1, and $3p^3+2p+2$.

4. Add $x^2+2xy+y^2$, $x^2-2xy+y^2$, x^2-4xy, and $4xy+y^2$.

5. Add $-23a^2b+41a^3c+56c^2b-15b^2c$, $-6a^2b+26a^2c+59c^2b-26b^2c$, $25a^2c-19b^2c+18c^2b$.

6. Add $\frac{5}{8}x^2-\frac{1}{3}xy-\frac{1}{4}y^2$, $-x^2-\frac{2}{3}xy+2y^2$, and $\frac{3}{8}x^2-xy-5y^2$.

7. A conductor collects during a trip 11 dollars, 20 quarters, 40 dimes, 62 nickels, and 18 cents. He pays out 2 dollars, 24 quarters, 15 dimes, 10 nickels, and 6 cents. How much does he make on the trip?

8. From $a^2+2ab+b^2$ subtract $a^2-2ab+b^2$.

9. From $x^3+3x^2y+3xy^2+y^3$ subtract $-3x^2y+3xy^2-3y^3$.

10. From $a^2b^2+a^4+b^4-3a^3b+4ab^3$ subtract $-a^4-b^4-a^2b^2-3a^3b-4ab^3$.

11. From $c^3-2a^2c+d^3-r^2$ subtract $c^3-a^2c-r^2$.

12. What is the meaning of $a-\big(b-(c+d)\big)$?

13. How would you remove the parentheses in problem 12?

In examples 14–17 remove the parentheses, without changing the values of the expressions:

14. $9x-\{5y-(6y+7z)-(7y-4z)\}$.

15. $29a-[15c-\{14a+23b-(17a+5b-38c)-(13c+18b-23a)\}]$.
16. $3a^2x-[2mb+\{a^2x-(-4s^2t+5mb)+s^2t\}]$.
17. $-(-(-(x-y)))$.

§ 47. Multiplication of Integral Algebraic Expressions

If in the sum the addends are all equal, the symbol, $\times$, or the dot, $(\cdot)$, is written between one of them and their number. Thus $a+a+a=3\times a$ or $3\cdot a$, $b+b+b\ldots\ldots$ (to a terms) $=a\cdot b$.

This method of forming out of two numbers as a and b a new number is called *multiplication.* The result of the multiplication is called the *product*, and the numbers which are multiplied are called the *factors.*

If the numbers are represented by letters the multiplication symbol may be omitted. Therefore, $a\times b$, $a\cdot b$, and ab have the same meaning.

Do 2×3, $2\cdot 3$, and 23 have the same meaning? May the multiplication sign between two arithmetic numbers be omitted? The multiplier 1 may or may not be omitted. Thus $1\cdot a=a$.

When the multiplier 1 is omitted it is said to be understood.

What is the meaning of $5\cdot 2$? $a\cdot 3$? $4\cdot b$? Show that as $2\cdot 3=3\cdot 2$, so $a\cdot b=b\cdot a$.

Illustrate, by taking any particular set of figures, that $abc=acb=cab=cba=bca$.

Prove the equality of the same products, assuming that $ab=ba$.

State the general principle just illustrated.

Multiply as indicated:

1. $4\times 13\times 15$.
2. $7\times 8\times 25$.
3. $4(2\times 3\times 6)$.
4. $5a\times 4$.
5. $6a\times 7\times b$.
6. $7(a\times b\times c\times d)$.
7. $7a\times 7b\times 7c\times 7d$.
8. $ax\times bx\times cx$.

Point out the difference between (6) and (7).

9. $4a \times 7b \times 25x$.

10. $3az \times 7bdz \times 8c \times 5y$.

11. $2abz \times 3bcz \times 4acy$.

12. $ab \times ab$.

13. $ab \times bc \times ac$.

14. $a^3 \times a^2$.

15. $b^7 b^5$.

16. $x^4 \times x^3$.

17. Show that $a^m \times a^n = a^{m+n}$, m and n denoting whole numbers. Formulate into words the *law of exponents in multiplication* expressed by this equation.

Multiply as indicated:

18. $a^2b^3c^4a^4b^4c^3$.

19. $2^6 \times 3^2$.

20. $5x^2y^3 \times 7 \times y^4z^5 \times 2x^6z$.

21. $1^3 + 3^3 + 5^3$.

22. $1^2 + 7^2 + 10^2$.

23. $6^2 + 8^2$.

24. $4^3 \times 5^6$.

25. $6^2 + 8^2$.

26. $2m^3t^2 \times 3mt^3$.

27. $8an^3x^2 \times 5ay^3$.

28. $m^3x^2 \times 3p^4xy^3$.

29. $\dfrac{3a^5m^2x^4}{4} \times \dfrac{2bm^3}{9}$.

30. $a \times -b$.

31. $2mt^3 \times -3mt^2$.

32. $a^2x^3 \times -a^3x^4$.

33. $am \times -am^2x$.

34. $2pq^2 \times -3p^4q^4$.

35. Write in the form of squares, cubes, and other powers the following expressions: $9p^2$, a^2x^2, $4m^2$, $121x^2$, $49z^2$, $81x^4$, $a^2b^2x^2y^2$, $121x^6y^6$, $36a^2b^2$, $a^2b^4c^6$, $49x^2y^4$, $8x^3$, $x^9y^6z^3$, $343x^3$, $27(x-y)^3$, $729(x+1)^3$, $64(a^2+b)^2$, $32x^5$, $729x^6$, $243x^5y^5z^5$, $216a^6$, $256m^8n^{16}$, $128a^7$.

To indicate that a sum, as $b+c$, is to be multiplied by a number a, the sum is inclosed in a parenthesis, thus $a(b+c)$.

$$\begin{aligned} \text{Since } a(b+c) &= (b+c)+(b+c)+ \ldots\ldots (\text{to } a \text{ terms}) \\ &= b+b+b+b \ldots\ldots (\text{to } a \text{ terms}) + c+c+c \\ &\qquad \ldots\ldots (\text{to } a \text{ terms}) \\ &= ab+ac. \end{aligned}$$

The product of $a(b+c)$ may be obtained by multiplying each term of the multiplicand by a and then adding the products.

Perform the indicated operations:

1. $a(b-c)$.	9. $a+b\times c$.
2. $(b+c)a$.	10. $(x+y)z+k$.
3. $3+(4+2)$.	11. $y+3(t+a)$.
4. $3+4\times 2$.	12. $8xy(8x-9y)$.
5. $a+b\times b$.	13. $16xyz(2xy-4yz+6xz)$.
6. $3+(4\times 2)$.	14. $a+(b+c)x$.
7. $3+4\times 2+7$.	15. $ax+by+(a-b)y$.
8. $(3+4)2+7$.	16. $5a(6a-7b)+8b(9a-10b)$.

17. $16z[(29z-a)+2(8a-14z)+7b]+4z(28b-3a)$.
18. $5a^2(5+6a^2-7a^4)$.
19. $17a^2(2a^2-3b^2-5a^2c)$.
20. $7m^2n^3(2m-3n-4mn+6m^3n)$.
21. $2ax(3x^2-4a^2x^3+5ax^4)$.

Problems and exercises:

1. Formulate into words the following equations:

$$a(b+c)=ab+ac\,; \qquad ab+ac=a(b+c)\,;$$
$$a(b-c)=ab-ac\,; \qquad ab-ac=a(b-c)\,.$$

2. Show similarly how to find the product of a monomial by any polynomial.

3. State whether these equations are true:

(1) $a(b+c-d-fr)=ab+ac-ad-afr$?

(2) $ma+mb-mc-md=m(a+b-c-d)$?

Give reasons.

4. Write in equivalent factored forms the following expressions:

(1) $5a-10b$;	(4) $15x^4-10x^3+5x^2$;
(2) $17x^2-289x^3$;	(5) $3m^5-12m^3n+6mn^4$;
(3) $16x^2-2abx$;	(6) $14x^2y^2z^2-7x^3y^3z^2+8xy^2z^2$;

(7) $60m^2n^3r^2-45m^3n^2r^3+90m^4n^2r^3+90m^4n^3r^2$.

5. Show that according to the definition of multiplication $(+a)\times(+b)=+ab$ and $(+a)\times(-b)=-ab$ (a and b being positive integers). Why have $(-a)\times(+b)$ and $(-a)\times(-b)$ no meaning according to the same definition?

DEFINITION. The last two cases can be included by enlarging the definition of multiplication: To multiply one algebraic number by another means to find the product of the multiplicand by the absolute value (value disregarding the sign) of the multiplier and then to prefix to this product the sign of the multiplier.

From this definition it follows that:

$$\begin{aligned}
&(1)\ (+a)\times(+b)=+(+ab)=+ab\,;\\
&(2)\ (+a)\times(-b)=+(-ab)=-ab\,;\\
&(3)\ (-a)\times(+b)=-(+ab)=-ab\,;\\
&(4)\ (-a)\times(-b)=-(-ab)=+ab\,.
\end{aligned}$$

Thus it appears that in multiplication *numbers having like signs give positive products; and numbers having unlike signs give negative products.*

Exercises:

6. Simplify:

(1) $a-7(b+c)$; (3) $-a^2\times(-a)^2$;

(2) $x-9(y-2)$; (4) $-a\times-a^2\times-a^3$;

(5) $-a\times(-a)^2\times(-a)^3$;

(6) $(pqr)\times(-p^2q^2r^2)\times(p^3q^3r^3)$;

(7) $14a+2b-2(3a-4b)+(a+b)$;

(8) $5a(6a-7b)-8b(9a-10b)$;

(9) $(25a^2-13a+11b)2a-(7a-15b)4a$;

(10) $[3a-7(5b+3c)]-a(3x-6b)+b(35x-6a)$;

(11) $7xy^2-[(18a+5b)x+5b(y-x)]z-2y(5b-76)$;

(12) $28a-5\{3[7a+5(5b-3a)+16(5a-3b)]-21(b-a)\}$.

7. Factor the following expressions:

(1) $4a+4b$; (5) $(a+b)x-(a+b)y$;

(2) $ax+bx$; (6) $16a^2x-24ax^2-14a^2+21ax$;

(3) $ax+ay-az$; (7) $ax+ay+bx+by$;

(4) $a(c+d)-b(c+d)$; (8) $ac-ad+bc-bd$;

(9) $56a^2xy+48ax^2y-16axy^2-7abc-6xbc+2ybc$.

8. Find the value of

$$(1)\ 67\cdot 7-23\cdot 7+7\cdot 35;$$
$$(2)\ 12\cdot 157-72+4\cdot 12-12\cdot 40.$$

Since $a\times 0=a(b-b)=ab-ab=0$; $\therefore a\times 0=0$.

Conversely, it is evident that a product is 0 if, and only if, at least one of its factors is 0. Hence, $(x-5)(x-8)=0$ if, and only if, $x-5=0$ or $x-8=0$; or if, and only if, $x=5$ or $x=8$.

From this example it is seen that, if in an equation all the terms be removed to one side, factoring that side may enable us to find the value of the unknown in the given equation. *The values of the unknown in an equation are called the roots of the equation.*

Exercises and problems:

1. Solve $12z^2=4z$.
2. Solve $5x^2-7x=0$.
3. Solve $(x-1)(x+1)(x-2)=0$.

4. The square of a number less five times the number equals zero. Find the number.

5. The square of a number added to twice that number less four times this second number is zero. What is the number?

6. The sum of a square and a rectangle, one of whose sides equals the side of the square, and the other side 2 inches less, equals the area of a rectangle with one side equal to the side of the given square and the other 5 inches greater.

7. Solve $ax+bx+cx=ac+bc+c^2$.

8. Multiply:

(1) $(a+2b)(3a+4b)$;
(2) $(-8x+5y)(-2x+3y)$;
(3) $(\frac{1}{3}s+\frac{4}{5}t)(\frac{3}{5}s-\frac{2}{3}t)$;
(4) $(2\frac{1}{2}m-3\frac{1}{3}m+5\frac{1}{4}p)(12m+24n-36p)$;
(5) $5\frac{2}{5}x+3\frac{1}{2}y-4z+7\frac{1}{2}u)(10x-20y+4z-8u)$ ·
(6) $(6x^2+2x+1)(x^2-x+1)$;

(7) $(x^2+4xy+y^2)(x^2+xy-y^2)$;
(8) $(x+y)(x^3+3x^2y+3xy^2+y^3)$;
(9) $(a^2-4ab-b^2)(a^3+2a^2b+ab^2+b^3)$;
(10) $(p^2+2pq+q^2)(p^2-2pq+q^2)$;
(11) $(x^3+3x^2y+4xy^2+5y^3)\ (x-y)$;
(12) $(3a+2b)(2ax-a^2-x^2)(bx-2a)$;
(13) $(3t^2-5v+2s)(s-2t+v)(3-s-t)$.

9. Show that the following five equations are true:

(1) $(a+b)^2=a^2+2ab+b^2$;
(2) $(a-b)^2=a^2-2ab+b^2$;
(3) $(a+b)(a-b)=a^2-b^2$;
(4) $(a+b)^3=a^3+3a^2b+3ab^2+b^3$;
(5) $(a-b)^3=a^3-3a^2b+3ab^2-3b^3$.

10. The length of the room exceeds its breadth by 5 ft. If the length had been increased by 3 ft. and the breadth diminished by 2 ft. the area would not have been altered. Find the dimensions of the room.

§ 48. Geometrical Representation of Special Forms of Products

1. Show from the figure that $xy+xz=x(y+z)$.

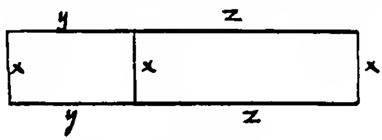

FIG. 57

2. Show from Fig. 58 that

$$ac+ad+bc+bd=(a+b)(c+d).$$

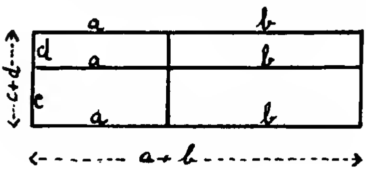

FIG. 58

3. Show from Fig. 59 that

$$a^2+2ab+b^2=(a+b)^2.$$

4. Show from Fig. 60 that

$$a^2-b^2=(a+b)(a-b).$$

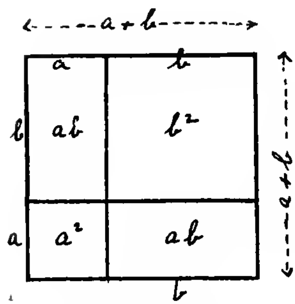

FIG. 59

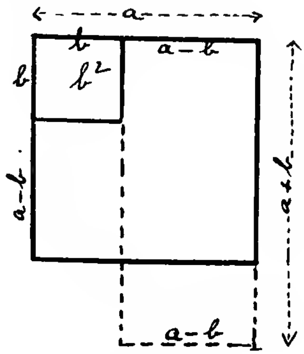

FIG. 60

5. Show by the aid of a divided rectangle that

$$(a+b)(c-d)=a\,c-a\,d+b\,c-b\,d.$$

6. Similarly work out the value of $(a-b)(c-d)$ with the aid of a divided rectangle.

7. What does the equation $(+a-b)(+c-d)=ac-ad-bc+bd$ become, when

(1) $b=0$, and $d=0$?
(2) $b=0$, " $c=0$?
(3) $a=0$, " $c=0$?

8. Compare each of the results of (7) with the expressions (1), (2), (3), and (4), p. 104, under problem 5.

9. Multiply by inspection:

(1) $(x+y)^2$;	(6) $(x+3)^2$;	(11) $(4a+7x)^2$;
(2) $(m+n)^2$;	(7) $(a-5)^2$;	(12) $(3m^4-2n)^2$;
(3) $(h+k)^2$;	(8) $(7-y)^2$;	(13) $(2x^2-3)^2$;
(4) $(a-p)^2$;	(9) $(9a-7b)^2$;	(14) $(2a^2x+3by^3)^2$;
(5) $(c-h)^2$;	(10) $(2x-3y)^2$;	(15) $[(a+b)+c]^2$;

(16) $[(a+b)-4c]^2$; (20) $(2a+3b)^3$;
(17) $(a+b)^2+(a-b)^2$; (21) $(3a-4b)^3$;
(18) $(4x+3y)^2-(2x-y)^2$; (22) $(4a^2b+5ab^2)^3$.
(19) $(3a+8b)^2+(4a+6b)^2-(5a-10b)^2$;

10. Transform the following differences into products:

(1) x^2-y^2; (4) $100a^4-81b^2$; (7) 67^2-33^2;
(2) $36x^2-25y^2$; (5) $144x^2y^2-81a^2y^2$; (8) 81^2-19^2;
(3) $16a^2-9b^2$; (6) 97^2-3^2; (9) 1017^2-17^2.

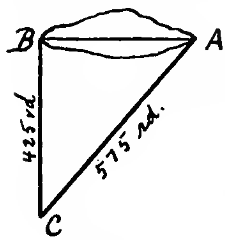

FIG. 61

11. A and B are stakes on opposite sides of a lake. AC=575 rd., BC=425 rd., angle ABC=90°, therefore $AB^2=AC^2-BC^2$. What is the distance across the lake?

12. The difference of the squares of two consecutive numbers is 25. Find the numbers.

13. Solve for x: $(x+b)^2-c^2=0$.

§ 49. Division of Integral Algebraic Expressions

To divide a number by a second number means to find a third number (quotient) which, when multiplied by the second (divisor), gives the first (dividend). According to this definition: *Quotient×Divisor=Dividend;* i. e., *Division is the reverse of multiplication.*

To indicate that a is to be divided by b, the symbols :, ÷, /, or –, are written between a and b; thus, $a \div b$, a/b, and $\frac{a}{b}$.

1. According to the definition find the quotient in $24m \div 4$; ab/a; $30ab/5$; a^3/a; a^9/a^5; x^3y^4/xy^2; $39a^5/13a$; $48x^3y^5/12xy^2$; $(ax+bx)/x$; $(ax-ay)/a$; $(7+7a)/7$; $(ab+a)/a$; $14/3$; $12a/5b$.

If an integer cannot be found which multiplied by b produces a, then, as in arithmetic, the indicated division, $\frac{a}{b}$, is called a *fraction*.

Since $a\times 0=0$, it follows from the definition of division that $0/a=0$.

2. What is the value of $0/0$? (Use definition of division and the equation $a\times 0=0$).

Since

$$\begin{array}{lll}(+a)\times(+b)=+(a\cdot b), & \text{therefore} & +(a\cdot b)\div(+b)=(+a);\\ (-a)\times(+b)=-(a\cdot b), & \text{“} & -(a\cdot b)\div(+b)=(-a);\\ (+a)\times(-b)=-(a\cdot b), & \text{“} & -(a\cdot b)\div(-b)=(+a);\\ (-a)\times(-b)=+(a\cdot b), & \text{“} & +(a\cdot b)\div(-b)=(-a).\end{array}$$

These results may be formulated in words thus: *In division, numbers, having like signs give positive quotients, and numbers having unlike signs give negative quotients.*

3. Find the quotients:

$$\frac{+20b}{-5a}, \quad \frac{-16x}{2y}, \quad \frac{-63a^2}{-7p^2}, \quad \frac{-35x^3}{-7p^2}.$$

4. What is the quotient of

$$\frac{b^4}{b^2}? \quad \frac{s^7}{s^4}? \quad \frac{x^{10}}{x^3}? \quad \frac{p^{18}}{p^7}? \quad \frac{x^n}{x^v}?$$

5. Can the quotient be obtained in this way with the following:

$$\frac{b^4}{a^2}? \quad \frac{p^{18}}{r^7}? \quad \frac{x^n}{y^v}?$$

6. How may the exponent of the quotient in each of the parts of Problem 4 be obtained from the exponent in the dividend and divisor?

7. Formulate into words *the law of exponents in division*, viz.:

$$\frac{a^m}{a^n}=a^{m-n}.$$

In the problems from which this law is obtained the exponent in the dividend is greater than in the divisor, i. e., $m>n$.

We will now consider cases where $m=n$, and where $m<n$.

§ 50. Cases in which $m=n$, and $m<n$.

1. *Case* $m=n$.

Suppose that the law is true, then $\frac{7^2}{7^2}=7^{2-2}=7^0$.

According to the definition of division $\frac{7^2}{7^2}=1$. Of the two answers obtained the first one, 7^0, has no meaning according to our present idea of the meaning of exponents; for we cannot take 7 zero times as a factor. If we agree that 7^0 shall mean 1, then the exponential law gives $\frac{7^2}{7^2}=7^{2-2}=7^0=1$, a result which agrees with that obtained from the definition of division.

In general, a^0 is defined by the equation $a^0=1$. Then the exponential law gives for $m=n$, $\frac{a^m}{a^m}=a^{m-m}=a^0=1$, which conforms to the definition of division.

2. Case $m<n$, or $n=m+k$.

Suppose that the exponential law holds. Then

$$\frac{a^m}{a^{m+k}}=a^{m-(m+k)}=a^{-k}.$$

Just as before, so here, the result has no meaning according to the definition of exponent. Show why it has not. It will be found advantageous to agree that a^{-k} be interpreted to mean $\frac{1}{a^k}$, i. e., $a^{-k}=\frac{1}{a^k}$.

Then both the exponential law and division lead to the same result. For, assuming that, as in arithmetic, dividend and divisor may be divided by the same factor, then

$$\frac{a^m}{a^{m+k}}=\frac{a^m}{a^m \times a^k}=\frac{1}{a^k}.$$

With these two new definitions, $a^0=1$, and $a^{-k}=\frac{1}{a^k}$, the

exponential law holds for any positive integral numbers, m and n.

Write the equivalents of the following expressions without using exponents:

$$2^{-3}, \quad 3^{-2}, \quad (\tfrac{1}{2})^{-4}, \quad (\tfrac{2}{5})^{-2}, \quad \frac{3^{-2}}{2^{-3}}.$$

§ 51. Exercises

Reduce the following to lowest terms by dividing out all common factors in numerators and denominators:

1. $\dfrac{378}{-63}$.
2. $\dfrac{3a^2bc}{-bc}$.
3. $\dfrac{-6x^2y}{-3xy}$.
4. $\dfrac{8x^2y^2}{2x}$.
5. $\dfrac{-25a^7bc^2}{5a^5bc}$.
6. $\dfrac{91a^5b^5mc^5}{-13ab^2mc^2}$.
7. $\dfrac{-a^5b^6c^7}{a^2b^2c^2}$.
8. $\dfrac{ax+bx}{x}$.
9. $\dfrac{ax-ay}{-a}$.
10. $\dfrac{ma+mb+mc}{m}$.
11. $\dfrac{as+bs-cs+ds}{s}$.
12. $\dfrac{3ar+3ab}{3axy}$.

§ 52. Division of a Polynomial by a Monomial

State a short way of dividing a polynomial by a monomial. Reduce the following to their lowest terms:

(1) $\dfrac{20a^3b-15a^2b^2+30ab^3}{5ab}$.

(2) $\dfrac{10x^2y-15x^3y^2+5x^3y}{-5xy}$.

(3) $\dfrac{36x^2y^4-42x^5y^6z}{-6x^2y}$.

(4) $\dfrac{35ab^8c^3-42a^3b^4c^3}{-7ab^3c}$.

(5) $\dfrac{2(a+b)^5-5(a+b)^3+2(a+b)^2}{(a+b)^2}$.

(6) $\dfrac{3(m-x)^4-2(m-x)^3-6(m-x)^2}{(m-x)^2}$.

§ 53. Division of a Polynomial by a Polynomial

Multiply (x^3-7x^2+2x+4) by $(4x^2+2x-3)$. The complete work consists of four steps, thus:

(1) $4x^2\times(x^3-7x^2+2x+4)=4x^5-28x^4+8x^3+16x^2$;

(2) $2x\times(x^3-7x^2+2x+4)=2x^4-14x^3+4x^2+8x$;

(3) $-3\times(x^3-7x^2+2x+4)=-3x^3+21x^2-6x-12$;

$\therefore$ (4) $(4x^2+2x-3)(x^3-7x^2+2x+4)=4x^5-26x^4-9x^3+41x^2+2x-12$.

Suppose next, that we have given for the dividend, the right member of equation (4), and the second factor of the left number as the divisor. The problem is to find the terms $4x^2$, $2x$, and -3 of the quotient.

It will be convenient to arrange dividend and divisor according to powers of the same letter, before beginning to divide. See the arrangement here given.

$$\begin{array}{rrrrrr|l}
4x^5 & -26x^4 & -9x^3 & +41x^2 & +2x & -12 & x^3-7x^2+2x+4 \\
\cline{7-7}
4x^5 & -28x^4 & +8x^3 & +16x^2 & & & 4x^2+2x-3 \\
\cline{1-4}
 & 2x^4 & -17x^3 & +25x^2 & +2x & & \\
 & 2x^4 & -14x^3 & +4x^2 & +8x & & \\
\cline{2-5}
 & & -3x^3 & +21x^2 & -6x & -12 & \\
 & & -3x^3 & +21x^2 & -6x & -12 & \\
\cline{3-6}
\end{array}$$

EXPLANATION.—First we are to find, $4x^2$, the first term of the quotient. $4x^5$ was found by multiplying $4x^2$ by x^3. Therefore $4x^2$ is found by dividing $4x^5$ by x^3.

Therefore, *the first term of the quotient is obtained by dividing the first term of the dividend by the first term of the divisor.*

Next, to find $2x$, the second term of the quotient.

Since $4x^2$ is now known we can form equation (1). Subtracting its right side from the dividend, the first term of the remainder is $2x^4$. $2x^4$ was found from $2x\times x^3$. Hence $2x^4\div x^3$ gives the second term of the quotient. Therefore *the second term of the quotient* is found by (1) multiplying the first term

of the quotient by the divisor; (2) subtracting this from the dividend; and (3) dividing the first term of the remainder by the first term of the divisor.

Similarly, it can be shown that any term of the quotient is found by, (1) forming the product of the preceding term by the divisor; (2) subtracting this from the last remainder; and (3) dividing the first term of the new remainder by the first term of the divisor.

Find the quotient of the first polynomial divided by the second, in each of the following problems:

1. $(x^2-2xy+y^2)\div(x-y)$.
2. $(a^3+3a^2b+3ab^2+b^3)\div(a+b)$.
3. $(6a^2+31ab+35b^2+12ac+42bc)\div(2a+7b)$.
4. $(x^2+7x+12)\div(x+3)$.
5. $(x^2-13x+30)\div(x-3)$.
6. $(x^2-\frac{5}{6}x+\frac{1}{9})\div(x-\frac{2}{3})$.
7. $(x^3-18\frac{2}{3}x+12)\div(x-\frac{1}{6})$.
8. $(3a^4+3a^2+5+3a+3a^5+5a^3)\div(1+a)$.
9. $(24m^2+42mn-62mp+15n^2-46np+35p^2)\div(6m+3n-5p)$.
10. $(a^3+b^3+c^3-3abc)\div(a+b+c)$.
11. $(2a^4+k^4-5a^3k-4ak^3+6a^2k^2)\div(k^2+a^2-ak)$.
12. $(a^4-16b^4)\div(a-2b)$.
13. $(x^5+y^5)\div(x+y)$.
14. $(x^3+y^3+z^3-3xyz)\div(x^2+y^2+z^2-xy-yz-zx)$.
15. $(6px^2+(914q-9p)xy+16vxz+10uvx-24ryz-21qy-15uvy)\div(3px+7qy+8vz+5uv)$.

CHAPTER XVI

FRACTIONS

§ 54. Fractions Having the Same Denominator

It was shown previously (p. 26, Ex. 6, (12), (13) that

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c};$$

and

$$\frac{a-b}{c}=\frac{a}{c}-\frac{b}{c}.$$

By interchanging the sides of each of these two equations, we get:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c},$$

and

$$\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

Similarly,

$$\frac{a}{m}+\frac{b}{m}+\frac{c}{m}+\frac{d}{m}=\frac{a+b+c+d}{m}.$$

Translate this last equation into words and obtain a rule for addition and subtraction of fractions that have a common denominator.

Combine each of the following expressions into a single fraction.

1. $\frac{12}{23}+\frac{8}{23}+\frac{15}{23}$.
2. $\frac{12}{23}+\frac{8}{23}-\frac{15}{23}$.
3. $\frac{x}{2}-\frac{y}{2}$.
4. $\frac{x}{a}-\frac{y}{a}$.
5. $\frac{a}{x}-\frac{b}{x}-\frac{c}{x}$.
6. $\frac{a+b}{2}+\frac{a-b}{2}$.
7. $\frac{x-y}{a}+\frac{2y}{a}$.
8. $\frac{c-ax}{4s}+\frac{c+3ax}{4s}$.
9. $\frac{3a}{5b}-\frac{3a-8b}{5b}$.
10. $\frac{10x}{17a}+\frac{12x}{17a}-\frac{5x}{17a}$.
11. Show that $-\frac{a}{b}=+\frac{-a}{b}=+\frac{a}{-b}$.

§ 55. Fractions Having Different Denominators

To show how to add or subtract fractions having different denominators, take, for example, $\frac{a}{b}+\frac{c}{d}$ and multiply it by bd, thus:

$$\left(\frac{a}{b}+\frac{c}{d}\right)bd=\frac{a}{b}\times bd+\frac{c}{d}\times bd\cdot \qquad \text{Why?}$$

$$=ad+bc. \qquad \text{Why?}$$

$$\left(\frac{a}{b}+\frac{c}{d}\right)bd=ad+bc. \qquad \text{Why?}$$

$$\therefore \quad \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}. \qquad \text{Why?}$$

1. In a similar way show that $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$.

2. Translate $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{ad}$ into words, and obtain a rule for addition and subtraction of fractions having different denominators.

3. According to this rule show that we must have

$$\frac{3}{x-1}+\frac{4}{x^2+1}=\frac{3(x^2+1)+4(x-1)}{(x-1)(x^2+1)}=\frac{3x^2+4x-1}{x^3-x^2-x-1}.$$

4. Express the following sums and differences as a single fraction:

(1) $\frac{1}{3}+\frac{1}{5}$;

(2) $\frac{1}{x}-\frac{1}{y}$;

(3) $\frac{x}{a}+\frac{x}{b}$;

(4) $\frac{a+3}{5}+\frac{a+5}{7}$;

(5) $\frac{a+x}{a-x}-\frac{a-x}{a+x}$;

(6) $\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}$;

(7) $\frac{1}{x+y}-\frac{1}{x-y}$;

(8) $\frac{x}{1-x^2}-\frac{x}{1+x^2}$;

(9) $\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}$.

In case the denominators of the fractions to be added have a common factor the fractions may first be reduced to equal fractions having for their denominators the least common multiple of all the given denominators, and then be added and simplified.

For example:

$$\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{10}=\frac{1\times 5}{2\times 5}+\frac{1\times 2}{5\times 2}+\tfrac{1}{10}=\tfrac{5}{10}+\tfrac{2}{10}+\tfrac{1}{10}=\tfrac{8}{10}=\tfrac{4}{5}.$$

5. Combine the following, each into a single fraction:

(1) $\frac{p}{2}+\frac{q}{5}+\frac{r}{10}$;

(2) $\frac{4b}{3}+\frac{5c}{6}+\frac{7d}{2}$;

(3) $\frac{5a}{12}-\frac{3a}{20}$;

(4) $\frac{a}{cx}+\frac{b}{cy}$;

(5) $\frac{1}{a}-\frac{y}{ax}$;

(6) $\frac{1}{c}-\frac{b}{ca}$;

(7) $\frac{x}{12ab}-\frac{z}{6b}$;

(8) $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$;

(9) $\frac{2}{xy}-\frac{3y^2}{xy^3}+\frac{xy+y^3}{x^2y^6}$;

(10) $\frac{a}{a-1}-\frac{ab}{a(a-1)}$;

(11) $\frac{1}{x-1}-\frac{1}{2(x-1)}$;

(12) $\frac{1}{2x-3y}+\frac{x+y}{4x^2-6xy}$;

(13) $\frac{b}{a+2b}+\frac{ab}{3ad+6bd}$;

(14) $\frac{a}{x}+b$;

(SUGGESTION.—Put $b=\frac{b}{1}$.)

(15) $\frac{x}{y}-a$;

(16) $5a+\frac{2a}{3}$;

(17) $11x-\frac{7x}{3}$;

(18) $\frac{3x}{4}-x$;

(19) $\frac{a}{x^2y}+\frac{b}{xy^2}$;

(20) $\frac{a+b}{x}+\frac{a-b}{3x}$;

(21) $\frac{a-2b}{3x}-\frac{4a-5b}{5x}$;

(22) $\frac{1}{a^3}-\frac{1}{a^2}+\frac{1}{a}$;

(23) $\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}$;

(24) $\frac{1}{x-y}-\frac{1}{y}$;

(25) $\frac{7a+3b-4c}{a}-\frac{2b+4a-3c}{b}+\frac{6a-7b-3c}{c}$;

(26) $\frac{5x-4y+3z}{x}+\frac{2x+3y-4z}{3y}-\frac{5x-6y-8z}{2z}$;

(27) $\frac{x}{x^2y}-\frac{y}{xz^2}+\frac{z}{x^2y}$;

(28) $\frac{a+b}{a}-\frac{a}{a-b}$;

(29) $\frac{(a+b)^2}{4ab}-1$;

(30) $\frac{x}{x^2-1}+\frac{x+3}{x-1}-\frac{x-2}{x+1}$.

6. Solve the following equations:

(1) $\frac{x}{2}+\frac{x}{3}-13+\frac{x}{4}=0$;

(2) $\frac{50}{x}+\frac{12}{x}-\frac{49}{10}=0$;

(3) $\frac{6x+7}{12}-\frac{x}{2}-\frac{3x-4}{4x-3}=0$;

(4) $\frac{y-1}{y+1}=1-\frac{1}{y}$;

(5) $\frac{3}{x+1}=\frac{2}{x-1}$;

(6) $\frac{7}{10}-\frac{1}{4y}=\frac{3}{5y}-1$.

7. The denominator of a fraction exceeds its numerator by 2; and if 1 be added to both, the resulting fraction will be be equal to $\frac{2}{3}$. What is the fraction?

§ 56. Multiplication of Fractions

How are fractions multiplied in arithmetic? Fractions in algebra are multiplied in the same way as in arithmetic thus:

$$\frac{y+x}{m+n}\times\frac{x^2-y^2}{\underset{2}{12}(m+n)}\times\frac{m+n}{m-n}\times\frac{6(m^2-n^2)}{x+y}$$

$$=\frac{(x^2-y^2)(m^2-n^2)}{2(m+n)(m-n)}=\frac{x^2-y^2}{2}.$$

EXERCISES

Perform the indicated operations:

1. $\frac{2}{5}\cdot\frac{3}{3}\cdot\frac{1}{8}$.

2. $\frac{68}{102}\cdot\frac{95}{133}$.

3. $\frac{1}{a}\cdot ac$.

4. $\frac{1}{a^2}\cdot a^3$.

5. $\frac{a}{b}\cdot\frac{1}{x}$.

6. $\frac{(6ab-4cd)}{12ab+8cd}\,(6ab+4cd)$.

7. $\frac{a}{c}\cdot\frac{c}{d}$.

8. $\frac{7xyz}{3bc}\cdot 9abc$.

9. $\frac{ab}{xy}\cdot\frac{yz}{bc}$.

10. $\frac{2ab}{3xy}\cdot\frac{5ax}{6by}$.

11. $\frac{15ab}{16xy}\cdot\frac{24yz}{25bc}$.

12. $\frac{3ab}{4xy}\cdot\frac{5bc}{6yz}\cdot\frac{7xz}{8ac}$.

13. $\frac{2a^2x}{3b^2y}\cdot\frac{6by^2}{7ax^2}\cdot\frac{5b}{4a}$.

14. $\frac{a^4b^4}{4x^2y^3}\cdot\frac{6x^4y^2}{5a^4b^2}$.

15. $\frac{a+b}{a-b}\cdot\frac{a^2-b^2}{a^2+b^2}$.

16. $\frac{27x}{8y+8x}\cdot\frac{x+y}{3}$.

17. $\frac{a}{a+b}\cdot\frac{b}{a-b}$.

18. $\frac{6(x-y)}{5xy^2}\cdot\frac{15x^2y^3}{8(x-y)}$.

19. $\frac{a^2-ab}{x^2-xy}\cdot\frac{x^2+xy}{a^2+ab}$.

20. $\frac{a^3-8b^3}{a^2-4b^2}\cdot\frac{a+2b}{a^2+2ab+b^2}$.

21. $\frac{a+1}{a-1}\cdot\frac{a+2}{a-2}\cdot\frac{a+3}{a-3}$.

22. $\frac{x^2-xy}{3a+3b}\cdot\frac{(a+b)^2}{(x-y)^2}$.

23. $\left(\frac{a}{x}-\frac{b}{y}\right)\left(\frac{a}{x}+\frac{b}{y}\right)$.

24. $\left(1-\frac{2ab}{a+b}\right)\left(\frac{a+b}{2ab}+1\right)$.

25. $\left(x+2y-\frac{5}{y}\right)\left(\frac{3y}{a+x}\right)$.

26. $\left(y+3-\frac{5}{y-3}\right)\left(2y+3-\frac{5}{2y-3}\right)$.

27. $\left(\frac{x^2}{a^2}-\frac{x}{a}+1\right)\left(\frac{x^2}{a^2}+\frac{x}{a}+1\right)$.

§ 57. Division of Fractions

The fraction obtained by interchanging numerator and denominator of a given fraction is called the *reciprocal* of the given fraction. Thus, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

Reciprocal fractions have the following property: *The product of a fraction and its reciprocal is unity.*

For $$\frac{a}{b}\cdot\frac{b}{a}=\frac{a\cdot b}{b\cdot a}=\frac{ab}{ab}=1\,.$$

Fractions are divided in algebra as in arithmetic. To prove this consider $a\div\frac{b}{c}$, where a may be either an integer, or a fraction.

$$a\div\frac{b}{c}=\left(a\div\frac{b}{c}\right)\frac{b}{c}\cdot\frac{c}{b}, \qquad \text{(Why?)}$$

$$=\left(a\div\frac{b}{c}\cdot\frac{b}{c}\right)\cdot\frac{c}{b}, \qquad \text{(Why?)}$$

$$=(a)\cdot\frac{c}{b}, \qquad \text{(Why?)}$$

Therefore $$a\div\frac{b}{c}=a\cdot\frac{c}{b}.$$

State in words a rule for multiplying a whole number by a fraction.

EXERCISES

Perform the indicated operations:

1. $x\div\frac{y}{x}$.
2. $ab\div\frac{xy}{z^2}$.
3. $12x^3\div\frac{16x^2}{7y}$.
4. $25a^6\div\frac{10a^2}{9x^2}$.
5. $18x^2y^2z^4\div\frac{15x^3y^3z}{ab}$.
6. $(-ab)\div\frac{3d}{5ab}$.
7. $(32x^3y^2z^2-40x^2y^3z^2+48x^2y^2z^3)\div\frac{8x^3y^3z^3}{a^3b^2}$.

8. $(7m^2pq-35mp^2q+21mpq^2)\div\dfrac{70m^2p^2q^2}{3r^2t^2}$.

9. $\dfrac{xyz}{3ab}\div\dfrac{x^2y^2z}{a^2b^2}$.

10. $\dfrac{a^2b^3z^4}{x^3y^2z}\div\dfrac{a^4b^3c^2}{xy^2z}$.

11. $\left(-\dfrac{15b^2}{16c}\cdot-\dfrac{27c^2}{105d}\right)\div\dfrac{abc}{28d^3}$.

12. $\left(\dfrac{m^2}{8n}\div\dfrac{15mpx^3}{27n^2y}\right)\cdot\dfrac{x^36p^3q^2}{m\cdot n}$.

13. $\left(\dfrac{x+1}{x}\div\dfrac{x^2-1}{x}\right)$.

14. $\dfrac{x^2-y^2}{a+b}\div\dfrac{x+y}{1}$.

15. $\dfrac{b-c}{b+c}\div\dfrac{1}{c+b}$.

16. $\dfrac{a+b}{x-y}\div\dfrac{a+b}{x^2-y^2}$.

17. $\dfrac{3(a+b)}{4(x-y)}\div\dfrac{6x(a+b)}{7a(x-y)}$.

18. $\dfrac{a^2-121}{a^2-4}\div\dfrac{a+11}{a+2}$.

19. $\dfrac{5a^2(x^2-y^2)}{6x^2(a-b)^2}\div\dfrac{10a(x+y)^2}{9x(a^2-b^2)}$.

20. $\dfrac{5a(3x-7y)+6b(7y-3x)}{(5a-6b)(2x+3y)}\div\dfrac{3x-7y}{4x^2-9y^2}$.

21. $\dfrac{11x^2(7m+5n)-(2x^2+16y^2)(7m+5n)}{(3x-4y)(25a^2-36b^2)}\div\dfrac{7m+5n}{5a-6b}$.

22. $\dfrac{210x^3y^2-45x^5y^3+105x^2y^4}{12p^4q^5+8p^3q^4+16q^3p^5}\div\dfrac{15x^2y^2}{4p^3q^3}$.

23. $\left(a+\dfrac{b}{c}\right)\div\left(\dfrac{a}{c}+\dfrac{b}{c^2}\right)$.

24. $\left(2x-\dfrac{x}{y}\right)\div(2y-1)$.

25. $\left(\dfrac{x}{x+1}-\dfrac{x-1}{x}\right)\div\left(\dfrac{x}{x+1}+\dfrac{x-1}{x}\right)$.

26. $\dfrac{1+x}{1-y^2}\div\dfrac{x^2-1}{y^2-1}$.

27. $\left(\dfrac{2a}{5b}\right)\div\left(-\dfrac{4x}{5y}\right)$.

28. $\dfrac{a-b}{b-a}$.

29. $\dfrac{x-1}{1-x}$.

30. 0. $\dfrac{3a-2b}{4b-6a}$.

31. 1. $\dfrac{10x-8y}{4y-5x}$.

32. 2. $\dfrac{2a\cdot(-3b)}{6ab}$.

33. $\dfrac{2\cdot(a-1)}{3\cdot(x-1)}\div\dfrac{4(1-a)}{5(x-1)}$.

34. $\dfrac{11(a-x)}{9(b-y)}\div\dfrac{7(x-a)}{6(y-b)}$.

35. $\dfrac{cy^2(m-r)}{fy(x-8)}\div\dfrac{c(m-r)}{f(x-8)}$.

CHAPTER XVII

EQUATIONS INVOLVING FACTORABLE FORMS

§ 58. A Monomial Factor

1. How long is each of the three parts of AB? How long is AB?

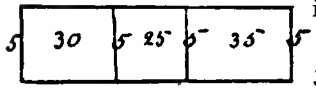

FIG. 62

2. Give a common factor of 30, 25, and 35. Give a factor of $30+25+35$, or 90.

3. Factor these sums and differences as the first one is factored:

(1) $15+21=3(5+7)$;
(2) $70+21=$?
(3) $60-35=$?
(4) $40-12=$?
(5) $12a+5a=$?
(6) $7x+13x=$?
(7) $14x+26x=$?
(8) $ab+ac=$?
(9) $xy-xz=$?
(10) $xy-x=$?
(11) $x^2y-x^2=$?
(12) $abc+aby=$?

4. Factor these sums and differences as the first one is factored:

(1) $12+9+6=3(4+3+2)$;
(2) $20+15-10=$?
(3) $18-6+10=$?
(4) $ab+ax+ad=$?
(5) $acx-acy+acz=$?
(6) $a^2b+a^2x+a^2=$?
(7) $3ab-3ac+6ad=$?
(8) $7c^2d-14c^2x+42c^2z=$?
(9) $6a^2+a^2d=$?
(10) $12cd-4cdx=$?
(11) $16x^2y+8xy^2=$?
(12) $7x^4y-21x^2y^4=$?

5. Make a rule for factoring sums and differences when the separate terms contain a monomial (one-termed) factor.

6. Solve the following equations by factoring the first member (side), dividing both members by a common factor, and then combining terms:

(1) $3x-18=9$;
(2) $2y+12=8$;
(3) $4y-4=16$;
(4) $7x+14=21$;
(5) $13x+26=39$;
(6) $9x-18=9$;
(7) $ax-a=3a$;
(8) $by-3b=2b$;
(9) $cz+ac=3ac$;
(10) $aby-abc=4abc$;
(11) $13ny+26n=39n$;
(12) $34x^2y-17xy=34xy$.

7. Given $5bx-acx=20b-4ac$; to find x.

Show by multiplying that $5bx-acx=(5b-ac)x$, and that $20b-4ac=4(5b-ac)$.

We may then write $(5b-ac)x=4(5b-ac)$. Why?

Dividing both sides by $5b-ac$, $x=4$.

8. Solve the following as 7 is solved:

(1) $3my+5dy=9m+15d$;
(2) $5rx-7x=25r-35$;
(3) $ax+bx=3ab+3b^2$;
(4) $cz-3bdz=5c-15bd$;
(5) $cx+dx-fx=7ac+7ad-7af$;
(6) $by-2dy+5ry=6bs-16bds+15brs$;
(7) $mrz-12az+b^2cz=8mrt-96at+8b^2ct$;
(8) $a^3x+5b^2x-c^5dx=7a^3s+35b^2s+7c^5ds$.

§ 59. A Binomial Factor

9. $(a+b)(x+y)=$? What are the factors of $ax+ay+bx+by$? How many terms of the product contain a as a factor? What numbers does a multiply?

How many terms of the product contain b as a factor? What numbers does b multiply?

REMARK: This product may also be shown by a divided rectangle.

10. $(a-b)(x-y)=$? What are the factors of $ax+ay-by-bx$?

11. $(a+b)(x-y)=$? What are the factors of $ax-ay+bx-by$?

12. $(a-b)(x-y)=?$ What are the factors of $ax-ay-bx+by$?

13. Examine the four products of 9, 10, 11, 12 and make a rule for finding the binomial factors of expressions having 4 terms, the first two and last two of which terms have a common factor.

14. Find what $x+y$ equals in the following:

(1) $ax+ay+bx+by=a+b$;
(2) $ax+ay+bx+by=3a+3b$;
(3) $ax+ay-bx-by=2a-2b$;
(4) $ax+ay-bx-by=7a-7b$;
(5) $ax+ay+bx+by=c(a+b)$;
(6) $ax+ay-bx-by=b(a-b)$.

15. To what is $a+b$ equal in these equations?

(1) $ax+ay+bx+by=3(x+y)$;
(2) $ax-ay+bx-by=c(x-y)$;
(3) $ax-ay+bx-by=12(x-y)$;
(4) $ac+ad+bc+bd=x(c+d)$.

16. To what is $a+x$ equal in these equations?

(1) $ab+bx+ay+xy=c(b+y)$;
(2) $ac-2ad+cx-2dx=cm-2dm$.

17. Find x in the following:

(1) $cx+dx+3c+3d=2(c+d)$;
(2) $bx+mx-5b-5m=3b+3m$;
(3) $5rx-5sx-2r+2s=8r-8s$;
(4) $cdx-b^2x-acd+ab^2=2acd-2ab^2$.

Squares and products of sums and differences.

18. $(a+b)(a+b)=?$ What are the factors of $a^2+2ab+b^2$?
19. $(a-b)(a-b)=?$ What are the factors of $a^2-2ab+b^2$?
20. $(a+b)(a-b)=?$ What are the factors of a^2-b^2?

21. Find the following products:

(1) $(x+y)(x+y)$;	(7) $(a+x)(a-x)$;
(2) $(x+y)(x-y)$;	(8) $(a+x)(a+x)$;
(3) $(x-y)(x-y)$;	(9) $(a-x)(a-x)$;
(4) $(b+c)(b-c)$;	(10) $(x+3)(x-3)$;
(5) $(b-c)(b-c)$;	(11) $(x+7)(x-7)$;
(6) $(b+c)(b+c)$;	(12) $(x-6)(x-6)$.

22. Find, by multiplying, the values of

(1) $(a+x)^2$; (3) $(c-y)^2$; (5) $(x+5)^2$; (7) $(y+2c)^2$;
(2) $(a-b)^2$; (4) $(r+s)^2$; (6) $(x-7)^2$; (8) $(p^2-2q)^2$.

23. Find, by multiplying, the values of

(1) $(a+b)(a-b)$; what are the factors of a^2-b^2?
(2) $(a+x)(a-x)$; " " " " " a^2-x^2?
(3) $(a+y)(a-y)$; " " " " " a^2-y^2?
(4) $(c+d)(c-d)$; " " " " " c^2-d^2?
(5) $(x+y)(x-y)$; " " " " " x^2-y^2?
(6) $(2a+3x)(2a-3x)$; what are the factors of $4a^2-9y^2$?
(7) $(x+5y)(x-5y)$; " " " " " x^2-25y^2?
(8) $(x+2cd)(x-2cd)$; " " " " " $x^2-4c^2d^2$?

24. Show that the following are perfect squares and give their square roots:

(1) $m^2+2mx+x^2$;	(6) $4a^2+4ac+c^2$;
(2) $c^2-2cd+d^2$;	(7) $4x^2-12xy+9y^2$;
(3) $a^2+2ad+d^2$;	(8) $a^2x^2+4axy+4y^2$;
(4) $x^2+2xy+y^2$;	(9) x^2-6x+9;
(5) $x^2-2xy+y^2$;	(10) $16r^2-24rs+9s^2$.

25. Show, by multiplying, that the square of any binomial sum or difference is the trinomial having two terms that are perfect squares and a third term which is plus or minus the double product of the square roots of the terms that are perfect squares.

26. Complete the following binomials into perfect trinomial squares by adding or subtracting *one* number:

(1) a^2+2ax;	(5) x^2+6x;	(9) $4a^2-2ab$;
(2) $-2ax+x^2$;	(6) x^2-6x;	(10) $9y^2+4x^2$;
(3) a^2+b^2;	(7) x^2+9;	(11) $9y^2+12xy$;
(4) a^2-2ac;	(8) $9-6x$;	(12) $4x^2-12xy$.

27. Find a value of the unknown, x or y, in each of these equations:

(1) $x^2+6x+9=2x+6$;	(5) $x^2-2ax+a^2=x-a$;
(2) $y^2-4y+4=6y-12$;	(6) $x^2-y^2=x-y$;
(3) $x^2+2xy+y^2=xy+y^2$;	(7) $y^2-b^2=y+b$;
(4) $x^2+10x+25=6x+30$;	(8) $x^2-c^2=x+c$.

28. (a) Given $5bx-acx=7a-2b$, to solve for x.

(b) Factoring first member, $x(5b-ac)=7a-2b$; for a factor of every term of a polynomial *is a factor of the polynomial.* The other factor is the quotient obtained by dividing the polynomial by this factor of every term.

Dividing both members of (b) by $(5b-ac)$: $x=\dfrac{7a-2b}{5b-ac}$.

29. (a) Given $ax+b^2=bx+a^2$. Solve for x, i. e., express x in terms of the other numbers in the equation.

(b) Transposing, $ax-bx=a^2-b^2$;
Factoring, $x(a-b)=(a+b)(a-b)$;
Dividing by $(a-b)$, $x=a+b$.

Many equations containing powers higher than the first power of the unknown number are readily solved by the use of factoring.

NOTE.—*The following work of this paragraph is to be developed by the teacher for the class.*

30. (a) Given $n^2-4n-12=0$; to solve for n:

(b) Factoring, $(n-6)(n+2)=0$;

(c) $n-6=0$, because 0 in place of $n-6$ in (b) satisfies the equation. (See Chap. XV, p. 105, under problem 8.)
Solving (c): $n=6$, one value of n.

(d) $n+2=0$, for same reason as in (c). $n=-2$ *another* value of n.

$\therefore$ 6 and -2 are the values of n.

31. (a) Given $x^3+x^2-6x=0$. To solve for x.
(b) Factoring, $x(x^2+x-6)=0$;
(c) Factoring again $x(x+3)(x-2)=0$;
(d) Therefore, $x=0$, one value of x;
(e) Also $x+3=0$ and $x=-3$, another value of x;
(f) Also $x-2=0$ and $x=2$, a third value of x.

How does the number of values of x seem to correspond with the exponent of the highest power of the unknown in the given equation?

32. (a) Given $y^3+y^2-6y=0$. To solve for y.
(b) Factoring, $y(y^2+y-6)=0$;
(c) Factoring further, $y(y+3)(y-2)=0$,
(d) Then $y=0$, one value of y;
(e) Also $y+3=0$, and $y=-3$, another value of y;
(f) Also $y-2=0$, and $y=2$, another value of y.

How is the number of values of y related to the exponent of the highest power of y in the equation?

33. (a) Given $x^3+x^2=4x+4$. Solve for x.
(b) Transposing $4x$ and 4: $x^3+x^2-4x-4=0$; Why?
(c) Factoring, $(x+1)(x+2)(x-2)=0$;

Therefore $x+1=0$, and $x=-1$;
$x+2=0$, and $x=-2$;
$x-2=0$, and $x=+2$.

Show by substitution that -1, -2, and 2 are values of x.

§ 60. Uses of Factoring in Solving Equations

The following problems illustrate the use of factoring in the solution of equations:

1. *Verbal Statement:* The difference of two numbers is one, and the difference of their squares is nine, find the numbers.

Symbolic statement:

(a) $x-y=1$;

(b) $x^2-y^2=9$, find x and y.

SOLUTION:

(c) Factoring x^2-y^2 in (b), $(x+y)(x-y)=9$;

(d) Dividing the equals in (c) by those in (a), or by substituting 1 for $(x-y)$ as per (a), gives $x+y=9$.

We now have $x-y=1$,

and $x+y=9$. Solve for x and y.

2. *Verbal statement:* The sum of the reciprocals of two numbers and also the difference of the squares of their reciprocals is five. Find the numbers.

Symbolic statement:

(a) $\frac{1}{x}+\frac{1}{y}=5$;

(b) $\frac{1}{x^2}-\frac{1}{y^2}=5$.

SOLUTION:

(c) Factoring, $\left(\frac{1}{x}+\frac{1}{y}\right)\left(\frac{1}{x}-\frac{1}{y}\right)=5$;

(d) Dividing the equals in (c) by (a), $\frac{1}{x}-\frac{1}{y}=1$;

(e) Adding (a) and (d) member by member:

$$\frac{2}{x}=6, \quad \therefore x=\tfrac{1}{3}.$$

(f) Subtracting (d) from (a): $\frac{2}{y}=4, \therefore y=\frac{1}{2}$.

3. *Verbal statement:* The sum of two numbers is 5, the sum of their cubes is 35. Find the numbers.

Symbolic statement:

(a) $x+y=5$;

(b) $x^3+y^3=35$.

SOLUTION:

(c) Factoring (b), $(x+y)(x^2-yx+y^2)=35$;

(d) Dividing equals (c) by equals (a): $x^2-xy+y^2=7$;

(e) Squaring equals in (a): $x^2+2xy+y^2=25$;

(f) Subtracting (d) from (e): $3xy=18$ and $xy=6$;

(g) Subtracting (f) from (d): $x^2-2xy+y^2=1$;

(h) Extract square root of equals: $x-y=\pm 1$;

(i) Add (a) and (h): $2x=6$ or 4, $\underline{x=3 \text{ or } 2}$;

(j) Subtract (h) from (a): $2y=4$ or 6, $\underline{y=2 \text{ or } 3}$.

Therefore the numbers are 3 and 2.

To verify, substitute the numbers found for x and y in (a) and (b) and state whether the results are correct equations.

4. Given $x^3-3x+2=0$; also $x^3-3x+2=(x-1)(x-1)(x+2)$: to solve for x and verify.

5. Given $x^3-2x^2-11x+12=0$; also $x^3-2x^2-11x+12=(x-1)(x+3)(x-4)$: to solve for x and verify.

6. Given $4x^4+9-37x^2=0$; also $4x^4-37x^2+9=\frac{1}{4}(2x-6)(2x+6)(2x-1)(2x+1)$: to solve for x and verify.

7. Given $\frac{2}{r}+\frac{3}{s}=3$, and $\frac{4}{r^2}-\frac{9}{s^2}=15$.

Notice $\frac{4}{r^2}-\frac{9}{s^2}=\left(\frac{2}{r}+\frac{3}{s}\right)\left(\frac{2}{r}-\frac{3}{s}\right)$. Solve for r and s and verify.

The student will supply the verbal statement for numbers 4 to 7. In practical life, problems are first stated in verbal form.

Multiply each side of the equations by the L. C. M. of the denominators.

8. $\dfrac{5x+1}{3}+\dfrac{19x+7}{9}-\dfrac{3x-1}{2}=\dfrac{7x-1}{6}$, solve for x.

9. $\dfrac{2x+1}{2x-1}-\dfrac{8}{4x^2-1}=\dfrac{2x-1}{2x+1}$, solve for x.

10. $\dfrac{4}{x+2}+\dfrac{7}{x+3}=\dfrac{37}{x^2+5x}+6$, solve for x

11. $\dfrac{x}{a-b}-\dfrac{5a}{a+b}=\dfrac{2bx}{a^2-b^2}$, solve for x.

CHAPTER XVIII

FRACTIONS INVOLVING FACTORABLE FORMS

A knowledge of factoring is often necessary in operations with fractions.

§ 61. Factoring in Reducing Fractions to Lowest Terms

1. In arithmetic $\frac{24}{32}$ is reduced by dividing numerator and denominator by the common factor 8, obtaining $\frac{3}{4}$, the same in value as $\frac{24}{32}$, but in lower terms.

2. An algebraic fraction is reduced in the same way: $\dfrac{2a^2bc}{5a^3b^2}$ can be reduced by dividing both numerator and denominator by the common factor a^2b, obtaining $\dfrac{2c}{5ab}$, having the same value in lower terms.

3. $\dfrac{a^2-b^2}{a^2-2ab+b^2}$ is reduced by dividing both terms by the factor $a-b$, obtaining $\dfrac{a+b}{a-b}$.

§ 62. Factoring in Adding and Subtracting Fractions

In the addition and subtraction of fractions, factors are used to find the lowest common denominator.

1. In arithmetic in finding the sum of $\frac{3}{20}$ and $\frac{7}{15}$, taking $3\times5\times4$, or 60, for the common denominator, instead of 20×15 or 300, we have

$$\frac{3\times3}{3\times5\times4}+\frac{4\times7}{3\times5\times4}=\frac{9}{60}+\frac{28}{60}=\frac{37}{60},\text{ the sum of }\frac{3}{20}\text{ and }\frac{7}{15}.$$

2. In $\frac{5}{8}+\frac{9}{16}-\frac{7}{24}$, taking $3\times2\times8$, or 48, instead of $8\times16\times24$, or 3,072, for the common denominator gives

$$\frac{2\times3\times5}{3\times2\times8}+\frac{3\times9}{3\times2\times8}-\frac{2\times7}{3\times2\times8}=\frac{30}{48}+\frac{27}{48}-\frac{14}{48}=\frac{43}{48}.$$

3. In algebra $\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}=\frac{a^2}{abc}+\frac{b^2}{abc}+\frac{c^2}{abc}=\frac{a^2+b^2+c^2}{abc}$.

4. $\frac{4x}{x^2-y^2}-\frac{2}{x-y}=\frac{4x}{(x-y)(x+y)}-\frac{2(x+y)}{(x-y)(x+y)}=\frac{2(x-y)}{x^2-y^2}$.

Show how $\frac{4x}{(x-y)(x+y)}$ is obtained from $\frac{4x}{x^2-y^2}$ and also how the succeeding fraction is obtained from the corresponding fraction.

§ 63. Factoring in Multiplying Fractions

1. Show that $\frac{2}{3}$ of $\frac{6}{7}$ of $\frac{5}{8}$ is reduced to $\frac{5}{14}$ by canceling the common factors 2, 3, and 2 from both the numerator and denominator.

2. In simplifying results of the multiplication of algebraic fractions also, we cancel *common factors* in numerator and denominator. Show that by canceling common factors

$$\frac{ab}{cd}\times\frac{ac}{bd}\times\frac{db}{ac} \text{ reduces to } \frac{ab}{cd}.$$

3. Show that in $\frac{2x}{7x+7y}\times\frac{x+y}{2y}$, by canceling the common factors, 2 and $x+y$, the product is $\frac{x}{7y}$.

4. In $\frac{a^2+2a-15}{a^2+8a-33}\times\frac{a^2+7a-44}{a^2+9a+20}$, factoring gives

$$\frac{(a+5)(a-3)}{(a+11)(a-3)}\times\frac{(a+11)(a-4)}{(a+5)(a+4)}.$$

Show that by canceling the common factors $a+11$, $a+5$, and $a-3$ we get $\frac{a-4}{a+4}$.

5. Show by multiplying that $(a+5)(a-3)=a^2+2a-15$; that $(a+11)(a-3)=a^2+8a-33$; that $(a+11)(a-4)=a^2+7a-44$; and that $(a+5)(a+4)=a^2+9a+20$.

CHAPTER XIX

FACTORING

The factors of a given number are the numbers which multiplied together will produce the given number. The process of factoring is therefore the inverse to the process of multiplication. In this chapter we consider only factors which are non-fractional (integral) and which do not involve radical signs (i. e., which are rational numbers). It is clear then that factors of a number will divide it without a remainder. A prime number is a number which has no factors except itself and unity. A prime factor is a prime number. A number has only one set of prime factors.

§ 64. Monomial Factors

1. *Problem:* Factor $3a^2b-12ab^2$.

EXPLANATION.—Since $3ab$ is a divisor of each term, it is a divisor of the whole expression, and dividing $3ab$ into the expression (each term in succession) we obtain the quotient $a-4b$. Then $3ab$ and $a-4b$ are the factors of $3a^2b-12ab^2$, or $3a^2b-12ab^2=3ab(a-4b)$.

Test by multiplication:

$$\begin{array}{r} a-4b \\ 3ab \\ \hline 3a^2b-12ab^2 \end{array}$$

PRINCIPLE.—*A number is equal to the product of all of its prime factors for all values of its letters.* Such an equality is called **an identity**.

By this principle, we can test, or verify the correctness of the factors. If the factors of $3a^2b-12ab^2$ are $3ab$ and $a-4$, then $3a^2b-12ab^2=3ab(a-4b)$ for all values of a and b (by the principle). The two expressions are therefore equal for the values, $a=3$ and $b=1$.

Test by substitution:

For $a=5$ and $b=1$;

$$3a^2b-12ab^2=3\times5^2\times1-12\times5\times1^2=15;$$

and $$3ab(a-4b)=3\times5\times1(5-4) \qquad =15.$$

Therefore the expression is correctly factored.

Verify by substituting other values for a and b.

2. Factor the following expressions and test results by both multiplication and substitution:

(1) $8x^3y^2+4x^2y^3$;
(2) $3x^2y^2-2xy-3xy^3$;
(3) $15a^3x-10a^3y+5a^3z$;
(4) $32a^3b^3-ab^6$;
(5) $6x^3y^2+3x^2y^3+3xy^4$;
(6) $8x^2y^2+16xyz-24x^2y^2z^2$;
(7) $2a^2b^2+a^3b^3+ab^4$;
(8) $8a^3b^2c^2-4a^2b^3c^3+a^2b^2c^3$;
(9) $15x^2+20xy+5y^2$;
(10) $16a^2b^2+48a^2b-16ab+8a$.

3. Reduce to lowest terms the fraction:

$$\frac{5a^2b-10ab^2}{15abc+20a^2b^2}.$$

4. Solve for x the equations:

(1) $ax+bx=ac+bc$; (3) $ax+bx=cx+d$;
(2) $abx-abd=cx-cd$; (4) $ax+bm=am+bx$.

5. Factor $3a+3b+5a+5b$.

SOLUTION:

$$\begin{aligned}3a+3b+5a+5b&=3(a+b)+5(a+b)\\&=(a+b)(3+5).\end{aligned}$$

The method of factoring used in this problem is called *the method of grouping*.

6. Factor $ac+bc+ad+bd$.

SOLUTION:

$$\begin{aligned} ac+bc+ad+bd &= c(a+b)+d(a+b) \\ &= (a+b)(c+d). \end{aligned}$$

Test by multiplication:

$$\begin{array}{l} c+d \\ a+b \\ \hline ac+ad+bc+bd\,. \end{array}$$

Test by substitution:

Let $a=1$; $b=2$; $c=3$; $d=4$.
Then $ac+bc+ad+bd=1\times3+2\times3+1\times4+2\times4$
$=3+6+4+8=21$,
and $(c+d)(a+b)=(3+4)(1+2)=7\times3=21$.
Therefore, $ac+bc+ad+bd=(c+d)(a+b)$.

7. Factor $14x^3-6x^2-21x+9$.
SOLUTION:

$$\begin{aligned} 14x^3-6x^2-21x+9 &= 2x^2(7x-3)-3(7x-3) \\ &= (2x^2-3)(7x-3). \end{aligned}$$

Test by both multiplication and by substitution.

8. Resolve into factors the following expressions and test results by both methods:

(1) $a^2-ax+ab-bd$; (6) $3ax+3ab+2x^2+2bx+b+x$;
(2) $x^6+5x^4+x^3+5x$; (7) $4x^3+4x-4x^2z-4z$;
(3) $6x^2-9x-10xy+15y$; (8) $1+r-r^2xy-r^3xy$;
(4) $2x^3+x^2+6x+3$; (9) x^2-x^3+1-x;
(5) $3ac+3ax-5c-5x$; (10) $(a+m)(c+n)-2n(a+m)$.

9. Reduce to lowest terms the following:

$$\frac{x^4-2x^3+7x+14}{2x^3-4x^2+6x-12}.$$

10. Find the product of the fractions

$$\frac{x^2-xy-6x+6y}{ax-bx-ay+by} \text{ and } \frac{5ac-5bc}{xy-6y}.$$

11. Solve for x:

(1) $ax+bx=ad+bd+ac+bc$;
(3) $3a^2bx+12bcd=4ab^2x+9acd$;
(3) $ax-bc-ac=ab+bx$.

§ 65. The Perfect Square

1. Factor $a^2+2ab+b^2$.

Method of grouping:

$$\begin{aligned} a^2+2ab+b^2 &= a^2+ab+ab+b^2 \\ &= a(a+b)+b(a+b) \\ &= (a+b)(a+b). \end{aligned}$$

Test by multiplication and by substitution that $(a+b)(a+b) = a^2+2ab+b^2$.

2. Factor $49x^3-154x^3y+121y^2$.

Method of grouping:

$$\begin{aligned} 49x^6-154x^3y+121y^2 &= 49x^6-77x^3y-77x^3y+121y^2 \\ &= 7x^3(7x^3-11y)-11y(7x^3-11y) \\ &= (7x^3-11y)(7x^3-11y) \end{aligned}$$

Test by multiplication and by substitution.

Method of inspection: We have learned (see pp. 107, 125, and 126) that when a binomial is multiplied by itself the result is a trinomial which consists of the square of the first term of the binomial plus or minus (according as the binomial is a sum, or a difference) twice the product of the first and second term of the binomial plus the square of the second term of the binomial.

Then any trinomial in which two terms are perfect squares (and positive) and the other term is plus or minus twice the product of the square roots of those two terms, is the square of the sum or difference of those two square roots according as the third term is plus or minus.

3. Add a third term to make perfect squares out of the following and factor:

(1) m^2+n^2;	(5) m^2+6mn;	(9) $16k^2+56ky^2$;
(2) m^2+2mn;	(6) n^2+8mn;	(10) r^2+3rs;
(3) $2mn+n^2$;	(7) $9a^2b^2+4c^2$;	(11) x^2+5x;
(4) m^2+4n^2;	(8) $9x^2+30k^3x$;	(12) x^2+bx.

4. Factor the following expressions, and test each result both by multiplication and substitution.

(1) $k^2+6kl+9l^2$;	(5) $c^2-16c+64$;
(2) $4m^2-12am+9a^2$;	(6) $x^{10}+30x^5+225$;
(3) $25+80r+64r^2$;	(7) $49-140n^2+100n^4$;
(4) $121a^2+198ay+81y^2$;	(8) $a^2b^2c^2+8abc+16$;

(9) $(u+v)^2+4t(u+v)+4t^2$;

(10) $m^2-2mn+n^2+6am-6an+9a^2$.

5. Factor the following by any method and test:

(1) $a^2k^3-2abk^3+b^2k^3$;

(2) $x^4y+x^2y^3-x^3y^2-xy^4$;

(3) ax^3+ax^2+ax+a;

(4) $(x+y)^2\cdot(x-y)-(x-y)^2\cdot(x+y)$;

(5) $a^2+b^2+c^2+2ab+2ac+2bc$;

(6) $x^2-6xy+9y^2-2xz+6yz+z^2$;

(7) $9a^2m+9a^2n+12am+12an+4m+4n$;

(8) $x^2-6xy+9y^2-2xz+6yz+z^2$.

6. Reduce the following fractions to lowest terms:

(1) $\dfrac{3a^2m+2b^2m}{9a^4+12a^2b^2+4b^4}$; (3) $\dfrac{ar^2s^2+br^2s^2+cr^2s^2}{a^2+b^2+c^2+2ab+2ac+2bc}$;

(2) $\dfrac{25c^2+10cd+d^2}{5ac+ad+5bc+bd}$; (4) $\dfrac{2ax-6x-2ay+6y}{3akx-9kx-3aky+9ky}$;

(5) $\dfrac{10klx^4+15klx^2+20kl}{2x^5+2x^4+3x^3+3x^2+4x+4}$.

7. Solve the following for the required numbers:

(1) $ax+3bx=a^2+6ab+9b^2$; x required.
(2) $2mr-10mr=15mr+4ns+6ns-3ny$; y required.
(3) $3ut+6vh^3+5wt=9uh^3+2vt+15wh^3$; t required.
(4) $7a^2b^2q+42a^2b^2c^2d^2=49a^4b^4+3c^2d^2q+9c^4d^4$; q required.
(5) $11a^2w-13b^2w-22a^2m+26b^2m=0$; w required; m required.

§ 66. The Difference of Two Squares

1. Factor a^2-b^2.

Method of grouping:

$$a^2-b^2=a^2-ab+ab-b^2 \qquad \text{(Why?)}$$
$$=a(a-b)+b(a-b) \qquad \text{(Why?)}$$
$$=(a+b)(a-b). \qquad \text{(Why?)}$$

Test in two ways.

Method by inspection: We know from the chapter on multiplication that when the sum of two numbers is multiplied by their difference, the result is the difference of their squares, viz.:

$$\begin{array}{lll} 2k & +3m & \\ 2k & -3m & \\ \hline 4k^2 & +6km & \\ & -6km & -9m^2 \\ \hline 4k^2 & & -9m^2 \end{array}$$

Therefore, the difference of two squares factors into the sum and difference of the square roots of the two squares.

2. Factor the following expressions and test results by both methods:

(1) x^2-y^2;
(2) $16k^2-25l^2$;
(3) r^4-s^4;
(4) $81m^4-16n^4r^4$;
(5) p^2q^2r-r;
(6) ax^4-100a;
(7) $16-25y^2$;
(8) $4a^2cd-25c^5d$;
(9) $(r+3s)^2-16t^2$;
(10) $b^2-(3a+2c)^2$;
(11) $16a^2-(2m-3n)^2$;
(12) $a^2-2ab+b^2-c^2$;
(13) $1-a^2-2ab-b^2$;
(14) $(3a-2b)^2-(2c-3d)^2$;

(15) $a^2+2a+2bc-b^2-c^2+1$;

(16) $9a^2-12ab+4b^2-16x^2-8xy-y^2$;

(17) $x^4+x^2y^2+y^4$;

(Suggestion: $x^4+x^2y^2+y^4=x^4+2x^2y^2+y^4-x^2y^2$).

(18) $a^4-7a^2b^2+b^4$;

(Suggestion: $a^4-7a^2b^2+b^4=a^4+2a^2b^2+b^4-9^2ab^2$).

(19) $16x^4-17x^2y^2+y^4$; (21) x^4+x^2+1 ;

(20) $25x^4+31x^2y^2+16y^4$; (22) $a^2x^8+a^2x^4+a^2$.

3. Reduce the following fractions to lowest terms:

(1) $\dfrac{mx^2-my^2}{ax-ay+bx-by}$;

(2) $\dfrac{a^2+4ab+4b^2-9c^2}{a^2+4b^2+9c^2+4ab-6ac-12bc}$;

(3) $\dfrac{k^3m^4+k^3n^2m^2+k^3n^4}{a^2m^2+a^2mn+a^2n^2}$;

(4) $\dfrac{1-a^2-2ab-b^2}{1+a^2+b^2+2a+2b+2ab}$;

(5) $\dfrac{x^2+ax+bx+ab}{(x^2-a^2)(x^2-b^2)}$.

4. In the following problems perform the indicated operations:

(1) $\dfrac{a^2+16a+64}{a^2-8a+16}\times\dfrac{a^2-16}{a^2-64}$;

(2) $\dfrac{4x^2-12xy+9y^2}{x^2-a^2-2ab-b^2}\times\dfrac{x^2+a^2+b^2+2ax+2bx+2ab}{4ax^2-9ay^2}$;

(3) $\dfrac{a^5-ab^4}{12a^2-75b^4}\div\dfrac{a^3+a^2b+ab^2+b^3}{24a^2b-120ab^3+150b^5}$;

(4) $\dfrac{2abx+3aby+2a^2b^2x+3a^2b^2y}{x^3+2abx^3+a^2b^2x^3}\div\dfrac{4x^2+12xy+9y^2}{mx^3+nx^2}$;

(5) $\dfrac{a^4-b^4}{a^2k^2+2abk^2+b^2k^2}\times\dfrac{mr+ms}{a^2-2ab+b^2}\div\dfrac{mc^2+mb^2}{k^2}$.

5. Solve the following equations for the required number:

(1) $ak+bk=a^2-b^2$. Solve for k.

(2) $a^2m-b^2m=an+bn$. Solve for m; solve for n.

(3) $k^4y+2k^2y-z=k^2z-y$. Solve for y; solve for z.

(4) $m^4-6m^2n^2+n^4=m^2r+2mnr-n^2r$. Solve for r.

(5) $a^2v+acv+3abv=a^2t-c^2t$. Solve for v; solve for t.

§ 67. The Trinomial of the Form x^2+ax+b

1. Factor $x^2+7x+12$.

Solution by separating and grouping:

$$\begin{aligned} x^2+7x+12 &= x^2+3x+4x+12 && \text{(Why?)} \\ &= x(x+3)+4(x+3) && \text{(Why?)} \\ &= (x+3)(x+4). && \text{(Why?)} \end{aligned}$$

Test in two ways.

2. Factor x^2-x-30.

Solution by grouping.

$$\begin{aligned} x^2-x-30 &= x^2-6x+5x-30 && \text{(Why?)} \\ &= x(x-6)+5(x-6) && \text{(Why?)} \\ &= (x-6)(x+5). && \text{(Why?)} \end{aligned}$$

Test in two ways.

3. Notice, in problem 1, viz.:

$$(x^2+7x+12)=(x+3)(x+4)$$

that the sum of the numbers $+3$ and $+4$ connected with x in the two factors is $+7$, which is the coefficient of x in the trinomial, and their product is $+12$, the term of the trinomial not containing x (i. e., the absolute term). Is this true in problem 2?

This is proved to be true in general as follows:

$$\begin{array}{l} x+a \\ \underline{x+b} \\ x^2+ax \\ \underline{\qquad\;\; +bx+ab} \\ x^2+ax+bx+ab=x^2+(a+b)x+ab. \end{array}$$

Therefore $x^2+(a+b)x+ab=(x+a)(x+b)$.

The sum of $+a$ and $+b=a+b$, the coefficient of x in the trinomial. The product of $+a$ and $+b=+ab$, the term of the trinomial not containing x (i.e., the absolute term). This gives us a method of factoring by inspection.

4. Factor $x^2+12x+35$ by inspection.

SOLUTION.—Find two factors of $+35$ whose sum is $+12$. They are $+5$ and $+7$. Then the factors of $x^2+12x+35$ are $(x+5)$ and $(x+7)$.

5. Factor $x^2-3x-40$ by inspection.

The factors of -40 whose sum is -3 are -8 and $+5$. Therefore $(x^2-3x-40)=(x-8)(x+5)$.

6. Factor the following expressions by inspection:

(1) x^2+3x+2;
(2) m^2+5m+6;
(3) $k^2+12kL+35L^2$;
(4) $y^2+14y+45$;
(5) $h^2-13h+40$;
(6) $a^2+8a-20$;
(7) $g^2+3g-180$;
(8) $b^2+19bc+48c^2$;
(9) q^6+q^3-42;
(10) $t^2-14t-51$;
(11) $r^2-2rs-323s^2$;
(12) $v^2-6vw-91w^2$;
(13) $a^2b^2-4abc^2-165c^4$;
(14) z^4-10z^2-24;
(15) $y^2-12y-85$;
(16) $d^2-7d-30$;
(17) p^2-p-90;
(18) $(a+b)^2-7(a+b)+10$;
(19) $a^2+2ab+b^2+3a+3b-10$;
(20) $a^2-6ab+9b^2+7ac-21bc-44c^2$.

7. Factor the following by any method and test results in one way:

(1) $a^2bc^3-2abc^3-8bc^3$;
(2) $b^2m+b^2n-12bm-12bn-45m-45n$;
(3) p^4-26p^2+25;
(4) p^4-16;
(5) $2ax+2ay-2az+2bx+2by-2bz$;
(6) $v^{16}-w^{16}$ into five factors;
(7) $m^2-2mn+n^2-c^2-d^2-2cd$;
(8) $8x^3-6x^2-28x+21$;

(9) $3k^2-6kl+3l^2+6km-6lm$;

(10) $x^2-4y^2+12yz-9z^2$.

8. Reduce the following fractions to lowest terms:

(1) $\dfrac{x^2+ax+bx+ab}{x^2+3ax+2a^2}$; (3) $\dfrac{2x+bx-3b-6}{ax-3a+bx-3b}$;

(2) $\dfrac{2ax+3bx+4a+6b}{x^2+x(b+2)+2b}$; (4) $\dfrac{m^2-12m+32}{m^2+2m-24}$.

§ 68. The Trinomial of the Form ax^2+bx+c.

1. Factor $6x^2+17x+12$.

Solution by separating and grouping:

$$\begin{aligned} 6x^2+17x+12 &= 6x^2+8x+9x+12 && \text{(Why ?)} \\ &= 2x(3x+4)+3(3x+4) && \text{(Why ?)} \\ &= (3x+4)(2x+3)\,. && \text{(Why ?)} \end{aligned}$$

Test in two ways.

2. Factor $8z^2-37z-15$.

Solution by separating and grouping:

$$\begin{aligned} 8z^2-37z-15 &= 8z^2-40z+3z-15 && \text{(Why ?)} \\ &= 8z(z-5)+3(z-5) && \text{(Why ?)} \\ &= (z-5)(8z+3)\,. && \text{(Why ?)} \end{aligned}$$

Test in two ways.

3. Factor $6a^2-7a-3$.

Solution by substitution: Multiply the expression by 6, and write the result in the following form:

$$(6a)^2-7\times 6a-18.$$

Put $z=6a$, and we have

$$z^2-7z-18,$$

which has two factors, $(z-9)$ and $(z+2)$. Then, since we have multiplied the expression by 6,

$$6a^2-7a-3=\frac{(z-9)(z+2)}{6}=\frac{(6a-9)(6a+2)}{3\cdot 2}=(2a-3)(3a+1)$$

4. Factor $5y^2+13y-6$.

Solution by substitution:

$$5y^2+13y-6=\frac{(5y)^2+13\times 5y-30}{5};$$
$$=\frac{z^2+13z-30}{5}\quad(\text{where } z=5y);$$
$$=\frac{(z+15)(z-2)}{5};$$
$$=\frac{(5y+15)(5y-2)}{5};$$
$$=(y+3)(5y-2).$$

5. Factor the following expressions, and test each solution:

(1) $2x^2+11x+12$;
(2) $8c^2+46c-12$;
(3) $3x^2-17x+10$;
(4) $8z^2-37z-15$;
(5) $5x^2-38x+21$;
(6) $11a^2-23ab+2b^2$;
(7) $7k^2+123k-54$;
(8) $12t^2+31st-15s^2$;
(9) $5m^2-29mn+36n^2$;
(10) $10r^2-23r-5$;
(11) $6b^2-29b+35$;
(12) $6f^2-f-77$;
(13) $6(a+b)^2-7(a+b)-3$;
(14) $4x^2+8xy+4y^2+13x+13y+3$;
(15) $3c^2-6cd+3d^2-2c+2d-5$.

6. Factor the following by any method and test the results:

(1) $2c^2xy-13cdxy+6d^2xy$;
(2) $3x^3+2x^2-9x-6$;
(3) x^4-2x^2+1;
(4) $6a^3-30a^2b+36ab^2$;
(5) $a^2-b^2-c^2-2bc+a+b+c$;
(6) a^2-b^2+a-b;
(7) $16x^4-81$;
(8) a^9-256a;
(9) $4a^4-9a^2+6a-1$;
(10) $a^2b^3-4ab^3-77b^3$.

7. Reduce the following fractions to lowest terms:

(1) $\dfrac{2a^2+17a+21}{3a^2+26a+35}$;

(2) $\dfrac{6d^2-5d-6}{8d^2-2d-15}$;

(4) $\dfrac{am^2-am-20a}{2m^2-7m-15}$;

(5) $\dfrac{15k^2+kl-2l^2}{9k^2+3kl-2l^2}$;

(3) $\dfrac{y^2z-8yz+15z}{2ay^2-13ay+21a}$; (6) $\dfrac{6x^2-bx-12b^2}{9x^2-3bx-12b^2}$.

8. In the following problems perform the indicated operations:

(1) $\dfrac{3}{18c^2+c-5}-\dfrac{4}{2c^2+7c-4}$;

(2) $\dfrac{10bx+3b^2+3x^2}{10bx-3b^2-3x^2}\div\dfrac{(3b+x)x}{(x-3b)b}$;

(3) $\dfrac{y^2+y-1}{2y^2-y-3}-\dfrac{y^2-y-1}{3y^2-y-4}$;

(4) $\dfrac{y^2-14y+24}{w^2+9w-36}\div\dfrac{(y^2+4y-12)\times(yw+5y)}{(w^2+2w-15)\times(yw+6w)}$.

9. Solve the following first degree equations for the desired number:

(1) $2am^2-5am+3m=bn^2+bn-2b$. Solve for a; for b.
(2) $15kx^2+kxy-6ky^2=9x^2+3xy-2y^2$. Solve for k.
(3) $tx^2-tx=20t+2x^2-7x-15$. Solve for t.
(4) $2a^2x+17ax+21x-3a^2y-26ay-35y=0$. Solve for x; for y.
(5) $mx^2+15m+13mx=8mx+2nx^2+21n$. Solve for m; for n.

10. Solve the following quadratic equations for the required number:

(1) $2x^2-13x+6=0$. Solve for x.

SOLUTION:

Factoring, $(2x-1)(x-6)=0$;
Then, by the principle stated on p. 105 under problem 8;
either $2x-1=0$, whence $x=\frac{1}{2}$,
or $x-6=0$, whence $x=6$.

and the two solutions of the quadratic equation

$2x^2-13x+6=0$, are $x=6$ and $x=\frac{1}{2}$.

(2) $6x^2+17x+12=0$. Solve for x.
(3) $6x^2=mx+35m^2$. Solve for m; for x.

(4) $36k^2=19kl+6l^2$. Solve for k; for l.
(5) $5y^2+13y=6$. Solve for y.
(6) $24m^2+15n^2=38mn$. Solve for m; for n.
(7) $2c^2+6d^2=13cd$. Solve for c; for d.
(8) $8a^2+14ab=15b^2$. Solve for a; for b.
(9) $14x^2+21y^2=55yx$. Solve for x; for y.
(10) $8k^2l^2=22klm+21m^2$. Solve for kl, for k, for l and m.

§ 69. The Sum or Difference of Two Cubes

1. By multiplying prove that

$$x^3+y^3=(x+y)(x^2-xy+y^2),$$

and that $x^3-y^3=(x-y)(x^2+xy+y^2)$.

Therefore, the sum of two cubes has for one factor the sum of the numbers that are cubed, and the other factor is the square of the first number of the factor minus the product of the first and second numbers plus the square of the second number.

Formulate a similar statement for the difference of cubes.

2. Factor the following expressions:

(1) $64a^3+27b^3$.

SOLUTION.

$$64a^3+27b^3=(4a+3b)(16a^2-12ab+9b^2).$$

The expression is the sum of two cubes $(4a)^3$ and $(3b)^3$, therefore one factor is the sum of the numbers cubed ($4a$ and $3b$), and the other factor is the square of the first number, $(4a)^2=16a^2$, of the first factor, minus the product of the first and second numbers, $-(4a\times 3b)=-12ab$, plus the square of the second number $(3b)^2=9b^2$.

(2) $8x^3-125y^3$.

SOLUTION:

$$8x^3-125y^3=(2x-5y)(4x^2+10xy+25y^2).$$

(3) m^3+27n^3;
(4) $8c^3-d^3$;
(5) $343-x^3$;
(6) t^3+64;
(7) $8x^{18}+27z^{12}$;
(8) $27u^9w^6+1$;
(9) $8v^{18}+27v^{12}$;
(10) $729a^6+216c^6$;
(11) $512c^3-27d^3$;
(12) k^3L^3+343;
(13) $(w+3)^3-a^3$;
(14) $(5m-n)^3+c^3$.

3. Factor the following and test:

(1) a^6-b^6;
(2) a^6+b^6;
(3) $a^{12}-b^{12}$;
(4) $a^{12}+b^{12}$;
(5) a^9+b^9;
(6) a^9-b^9;
(7) a^3+b^3+a+b;
(8) a^3-b^3+a-b;
(9) $x^3-y^3-3xy(x-y)$;
(10) $a^4-a^3b+ab^3-b^4$.

4. In the following problems perform the indicated operations and reduce to lowest terms:

(1) $\dfrac{x^3-27}{x^2+2x-15}$;

(2) $\dfrac{1}{x-2a}+\dfrac{a^2}{x^3-8a^3}-\dfrac{x+a}{x^2+2ax+4a^2}$;

(3) $\dfrac{a^4+a^2b^2+b^4}{a^6-b^6}\times\dfrac{a+b}{a^3+b^3}\times\dfrac{a^2-b^2}{a}$;

(4) $\dfrac{x^2-xy+y^2}{x^2+xy+y^2}\times\dfrac{x^3-y^3}{x^3+y^3}\div\dfrac{(y-x)^2}{(x+y)^2}$;

(5) $\dfrac{x^3-8y^3}{x^2-xy}\times\dfrac{x^2-xy+y^2}{x^2+2xy+4y^2}\times\dfrac{x^3-xy^2}{x^3+y^3}$.

§ 70. The Factor Theorem

1. Divide ax^2+bx+c by $x-r$.

$$
\begin{array}{r|l}
x-r)\,ax^2+bx+c & (ax+b+ar \\
\underline{ax^2-axr} & \\
bx+arx & \\
\underline{bx-br} & \\
arx+br+c & \\
\underline{arx-ar^2} & \\
\text{Remainder,}\qquad ar^2+br+c. &
\end{array}
$$

2. What is the remainder obtained by dividing kx^3+lx^2+mx+n by $x-t$?

3. What is the remainder obtained by dividing x^2+px+q by $x+k$?

4. What is the remainder obtained by dividing kx^3+lx^2+mx+n by $x+a$?

5. We find that in these four problems the remainder in each case is the expression obtained by replacing x in the dividend by the negative of the number connected with x in the divisor. Verify this in each problem by substituting.

In problem 1, if ar^2+br+c had turned out to be zero, there would be no remainder and $x-r$ would be an exact divisor or a factor of ax^2+bx+c.

6. If x^3-5x^2+2x+4 be divided by $x-3$ what remainder would be expected from the above results?

Verify your answer by actual division.

7. If $3x^2-5x^2+4x+1$ is divided by $x+2$, by substitution what is the remainder? Verify by division.

8. Show that x^3-3x^2+2x-6 has a factor $x-3$ by using the above principle (without dividing).

9. The proof will now be given of the theorem we have been using, i. e., that, if in a given expression containing x (call it for brevity an expression in x), r is put for x, the expression in x reduces to the remainder R, obtained by dividing the expression in x by $x-r$.

PRINCIPLE.—*In division there is always a divisor d, a dividend D, a quotient Q, and a remainder R (sometimes zero), and the following relation connects them.*

$$D=Q\times d+R.$$

Then, if the expression in x is divided by $x-r$, by this principle we have the expression in x $=Q(x-r)+R$.

Substitute on both sides, for x, the number r, then expression in r $=Q(r-r)+R=Q\times 0+R=R$.

Therefore, R, the remainder found by dividing the expression in x by $x-r$, equals expression in r, i.e., the given expression with r put in place of x.

10. Factor:

(1) $x^3-5x^2-2x+24$;

(2) x^3+8;

(3) $x^3-7x^2+7x+15$;

(4) x^3-1;

(5) $x^4-3x^3-21x^2+43x+60$;

(6) $x^4-15x^2+10x+24$;

(7) $x^3+2ax^2+5a^2x+4a^3$;

(8) y^3+64.

11. Solve the following equations:

(1) $x^3+3x^2-13x-15=0$.

SOLUTION: $x=-1$, reduces the expression on the left to zero, then $x-(-1)$, or $x+1$ is a factor.

By dividing we get:

$$\begin{aligned} x^3+3x^2-13x-15 &= (x+1)(x^2+2x-15) \\ &= (x+1)(x-3)(x+5); \end{aligned}$$

Then $(x+1)(x-3)(x+5)=0$,

Whence $x=-1, 3, -5$. Why?

(2) $k^3+2k^2+4k+8=0$. Solve for k.

(3) $m^3-6m^2+11m-6=0$. Solve for m.

(4) $4y^4+32y^3+83y^2+76y+21=0$. Solve for y.

(5) $a^4-11a^3+44a^2-76a+48=0$. Solve for a.

CHAPTER XX

RATIO, PROPORTION AND SIMILARITY

§ 71. Examples and Definition

The ratio of 6 to 3 is $\frac{6}{3}$, or 2; of 3 to 4 is $\frac{3}{4}$; of a to b is $\frac{a}{b}$.

The ratio of 6 to 3 is sometimes written 6:3; of 3 to 4, 3:4; and of a to b, $a : b$.

DEFINITION.—*The ratio of any number to another number is the quotient found by dividing the first number by the second.* Thus $\frac{2}{3}$ is the ratio of 2 to 3. Any fraction may be regarded as an expression of the ratio of its numerator to its denominator.

In all problems and exercises, answer all you can orally. Use pencil and paper only when necessary.

§ 72. Exercises and Problems

1. Write in two ways the following ratios:

(1) 5 to 20;	(7) 10 to 50;	(13) $a+b$ to c;
(2) 20 to 5;	(8) x to y;	(14) $x+y$ to $x-y$;
(3) 15 to 20;	(9) c to d;	(15) $a-b$ to $c+d$;
(4) 18 to 25;	(10) d to c;	(16) $ax+ay$ to a;
(5) 25 to 18;	(11) m to r;	(17) $3x+2$ to ab;
(6) 50 to 10;	(12) y to x;	(18) $a+b$ to x.

2. A rectangle is 6′ by 18′ and another 7′ by 18′. What is the ratio of their areas? Of their lengths? Of their widths?

3. One triangle has a base of 24′ and an altitude of 10′; another has a base of 12′ and an altitude of 10′. What is the ratio of their areas? Of their bases? Of their altitudes?

4. What is the ratio of the area of a rectangle 12′×20′ to that of a triangle 12′ high by 20′ long? What is the ratio of their lengths? Of their heights?

5. How do two numbers whose ratio is 1 compare in size?

6. What is the ratio of areas of a triangle and a rectangle whose bases are b and altitudes a?

7. What is the ratio of the cost of 5 yd. of silk at $1.50 to 50 yd. of cotton at $12\frac{1}{2}$ cents?

8. What is the ratio of one yard to one foot? Of 1 yd. to 1 in.? Of 1 yd. to 6 in.? Of 3 yd. to 3 in.? Of 3 yd. to 3 feet?

9. What is the ratio of 1 lb. to 1 oz.? Of 1 oz. to 5 lb.? Of 1 ton to 500 lb.? Of 5 lb. to 5 ounces?

10. What is the ratio of 1 mi. to 1 yd.? Of 1 mi. to 1 ft.? Of 1 mi. to 880 ft.? of 1 mi. to 880 inches?

Problems 8, 9, 10 illustrate that magnitudes must be expressed in the same unit before their ratio can be expressed as a single number.

11. How does the ratio of the areas of the rectangles of problem 2 compare with the ratio of the widths? How do the bases compare?

12. The altitude of one rectangle is $10''$ and the base is $4'$; the altitude of another rectangle is $20''$ and the base is $4'$. What is the ratio of their areas? Of their altitudes?

13. Two rectangles have bases $20'$ and $25'$ and both have an altitude of $15'$. What is the ratio of their areas? Of their bases?

14. Two rectangles have altitudes of $8'$ and $15'$ and both have a base of $18'$. What is the ratio of their areas? Of their altitudes?

15. The dimensions of one rectangle are a and b and of another are a and c. What is the ratio of their areas? Of their unequal dimensions?

16. How does the ratio of the areas of the rectangles having equal bases compare with the ratio of their altitudes? How does the ratio of the areas of the rectangles having equal altitudes compare with the ratio of their bases?

17. Compare the ratio of the areas and of altitudes of triangles with equal bases. Compare the ratios of areas and of bases of triangles with equal altitudes. Compare the ratios of areas and of altitudes of parallelograms with equal bases. Compare the ratios of areas and of bases of parallelograms with equal altitudes.

DEFINITION.—An equation of ratios is called a *proportion.* For example: $\frac{4}{6}=\frac{2}{3}$; and $\frac{a}{b}=\frac{ac}{bc}$, and $\frac{a}{b}=\frac{c}{d}$ are all called proportions and are sometimes written thus: $4:6=2:3$, $a:b=ac:bc$, and $a:b=c:d$. The last is read "a is to b as c is to d." Read the other two. Numbers that form a proportion are said to be *proportional.*

18. Show that (1) areas of rectangles are proportional to their bases, if their altitudes are equal, (2) areas of rectangles and their altitudes are proportional, if their bases are equal, (3) areas of rectangles are proportional to the products of their bases and altitudes, (4) areas of triangles are proportional to the products of their bases and altitudes.

19. Is $2:5=6:15$ or $\frac{2}{5}=\frac{6}{15}$, a true proportion? Why? Is $2:7=8:25$ a true proportion? Give reason for your answer.

DEFINITION.—The first and last terms of a proportion are called *extremes;* the second and third, the *means.*

20. Compare the product of the extremes with the product of the means in $2:5=6:15$; in $3:7=6:14$; in $20:2=10:1$; in $12:3=4:1$. What do you find true of the products?

21. For which of the following expressions is the product of the means equal to the product of the extremes:

(1) $1:3=4:12$;	(6) $2:3=2a:3a$;
(2) $3:4=6:12$;	(7) $8:8a=3:3a$;
(3) $2:3=8:11$;	(8) $x:y=4x:4y$;
(4) $5:6=10:12$;	(9) $3a:3x=6a:3x$;
(5) $8:3=15:6$;	(10) $x:3x=1:3$.

Is there any true proportion in this list for which the equality of products does not hold?

Is there any expression in the list that is not a true proportion, for which the product of the first and fourth numbers is equal to the product of the second and third?

PRINCIPLE.—*In any proportion the product of the means equals the product of the extremes.* This is a convenient test of proportionality.

22. By this test tell what expressions in problem 21 are correct proportions.

23. From $4\times6=3\times8$, write a proportion in as many ways as you can, using only the numbers 4, 6, 3, and 8.

24. Show by using arithmetical numbers that, if the product of two numbers (as 4×8) equals the product of two other numbers (as 2×16), either pair, that is, either the 4 and 8, or the 2 and 16, may be made the extremes and the other pair the means of a proportion.

25. Make a proportion in at least four ways from each of the following equations, and show by the law for testing proportionality that your proportions are true:

(1) $2\times10=4\times5$;	(6) $(a+b)^2=mn$;
(2) $4\times9=3\times12$;	(7) $m^2-r^2=a^2-x^2$;
(3) $5\times8=4\times10$;	(8) $x^2-3x+2=a^2+4a-5$;
(4) $3a=4b$;	(9) $c^2-d^2=x^2+2xy+y^2$;
(5) $ab=xy$;	(10) $a^2-c^2=x^2-r^2$.

26. Prove that if $a:b=b:c$ then $b:a=c:b$.

DEFINITION.—In a proportion having the same number for both the second and third terms, this number is called a *mean proportional* between the other two terms. Of course, the mean proportional may be made the first and fourth terms of a proportion.

27. Show that the mean proportional may be made the first, or fourth, term of a correct proportion with the other two numbers.

When there is a mean proportional, the fourth term is called a third proportional to the other two numbers.

28. Show that in $2:5=6:15$ the 5 and 6 may change places giving a correct proportion. Show also that the 2 and 15 may be interchanged without destroying the proportionality.

DEFINITION.—*A proportion made from another proportion by interchanging either the means or the extremes is said to be formed by alternation.* In the same way the first proportion is said to be "taken by alternation."

29. Take the following proportions by *alternation* and prove (by the test principle) that you obtain true proportions:

(1) $8:11=24:33$;	(6) $a:b=c:d$;
(2) $7:12=35:60$;	(7) $c:d=x:y$;
(3) $3:19=12:76$;	(8) $a:x=c:y$;
(4) $32:5=128:20$;	(9) $a+b:x=c+d:y$;
(5) $17:8=85:40$;	(10) $3a+1:r=a+3:7$.

§ 73. Similar Figures and Proportionality

1. Draw a triangle having two of its sides 10″ and 12″ long and including any convenient angle between them. Call the third side the base. Through a point on the 10″ side and 5″ from the vertex of the included angle draw a parallel to the base. Measure the distance on the 12″ side from the vertex of the included angle to the crossing-point of the parallel. How does the ratio of the parts of the sides cut off by the parallel compare with the ratio of the sides 10 and 12?

2. Draw a parallel to the base through a point of the 10″ side $2\frac{1}{2}$″ from the vertex and measure the distance from the vertex to the crossing-point of the $2\frac{1}{2}$″ parallel with the 12″ side. Compare the ratio of the parts of the sides with the ratio of the sides themselves (10 : 12).

3. Compare the ratios of the corresponding parts of the sides made by parallels to the base, through a point of the 10″ side 3″ from the vertex; 6″ from the vertex; $1\frac{1}{4}$″ from the vertex; $7\frac{1}{2}$″ from the vertex; $8\frac{3}{4}$″ from the vertex.

4. Draw two triangles having the same shape but different sizes. Measure two corresponding pairs of sides in each and compare their ratios. Measure another pair of corresponding sides and compare their ratios. What seems to be true of the ratios of corresponding pairs of sides of triangles having the same shape (similar triangles)?

5. Draw two squares of different sizes and compare the ratio of the lengths of their corresponding sides.

6. Draw two rectangles with corresponding sides proportional but differing in size and compare the shapes of the two rectangles.

CONCLUSION.—*In all plane figures of the same shape (similar figures) and having the same number of sides, corresponding sides are proportional.*

7. Is $3:7=9:21$ a proportion? Why? Add the second term to the first and the fourth to the third and ascertain whether the first sum is to seven as the second sum is to 21? Also whether the first sum is to 3 as the second sum is to 9.

8. Make two triangles, one with sides 3, 5, and 7 and the other with sides 9, 15, and 21. How do the triangles resemble each other in shape, or form? In size? How does the ratio of the first two sides of the first triangle compare with the ratio of the two corresponding sides of the second triangle? How does the ratio of any pair of sides of one of the triangles compare with the ratio of the corresponding sides of the other?

9. Show by drawing triangles and extending sides what is shown with arithmetical numbers in problem 7?

DEFINITION.—When a proportion is made from a given proportion by placing the sum of the two terms of each ratio in place of the antecedent (first term) of each ratio, or in place of the consequent (last term) of each ratio, the given proportion is said to be *taken by composition.* If the difference instead of the sum of the terms of each ratio is used in this way, the given proportion is said to be *taken by division.*

10. Write a proportion, first, by *composition*, then by *division* from the following, and prove that the proportions you write are correct:

(1) $8:3=16:6$;	(6) $a-b:b=c-d:d$;
(2) $12:10=36:30$;	(7) $a+3:a=z+6:6$;
(3) $5:2=30:12$;	(8) $a:b=c:d$;
(4) $15:12=10:8$;	(9) $a:c=x:2$;
(5) $4a:a=8b:b$;	(10) $x:a=c:d$.

§ 74. Problems and Applications

Find the value of the letter in each of the following proportions:

1. $x:2=12:1$.	6. $5-x:3=12:2$.
2. $8:x=24:2$.	7. $a+8:9=36:6$.
3. $7:35=a:25$.	8. $15:3x+1=6:3$.
4. $9:11=54:x$.	9. $28:7=4x-2:14$.
5. $18:3=z:1$.	10. $x-5:5=4:9$.

11. If 17 acres of land cost \$850 what would 51 acres cost, at the same price?

12. The force f required to hold a load, L, from rolling down a slope, as AC, is found by the proportion $f:L=\text{BC}:\text{AC}$; that is, the force is to the load as the height BC of the slope is to its length AC. Find the force that is needed to prevent a load of 2,000 lb. from rolling down a slope that is 1,200 ft. long and 100 ft. high.

13. Find the force needed to prevent rolling downward of the following loads, the lengths and heights of the slope, as indicated:

(1) $L=1{,}500$,	AC = 2,000 ft.,	BC = 200 ft.;
(2) $L=2{,}800$,	AC = 3,563 ft.,	BC = 509 ft.;
(3) $L=5{,}000$,	AC = 1 mi.,	BC = 528 ft.;
(4) $L=30$ tons,	AC = 2 mi.,	BC = 528 ft.;
(5) $L=150T$,	AC = 1 mi.,	BC = 158.4 ft.;
(6) $L=300T$,	AC = $\frac{1}{2}$ mi.,	BC = 792 feet.

14. The weight of sheet metal is proportional to the area of the sheet. If a rectangle of sheet zinc 18 in.×24 in. weighs $7\frac{1}{2}$ oz., how much does a rectangle 24 ft.×15 ft. of the same sheet weigh?

15. Fifteen sq. ft. of sheet copper weighs $2\frac{1}{2}$ lb. How much will a piece of the same sheet 8 ft. by 24 ft. weigh?

16. If t_1 denote the time it takes for a pendulum of length L_1 to vibrate once, and t_2 and L_2 denote similar numbers for another pendulum, a law of vibration is (1) $t_1 : t_2 = \sqrt{L_1} : \sqrt{L_2}$, which gives (2) $t_1^2 : t_2^2 = L_1 : L_2$. If a pendulum 36 in. long vibrates in .5 sec., how long will it take a pendulum 49 in. long to vibrate? 25 in. long? 64 in. long? 81 in. long?

17. If a pendulum 39 in. long vibrates in 1 second, how long will it take a pendulum 13 ft. long to vibrate?

18. State in words the laws of problem 15, (1) and (2), for vibrating pendulums.

19. The areas of similar triangles are as the squares of corresponding sides. Show by means of a figure that this is true.

20. The side of a triangle is 8 in. long and its area is 24 sq. in. The corresponding side of a similar triangle is 12 in. What is the area of the second triangle?

21. The areas of all similar figures are proportional to the squares of corresponding sides. Show by a figure what this means.

22. If two parallelograms are similar, and a side and the area of one are 6 ft. and 45 sq. ft., respectively, and the side of the second parallelogram which corresponds to the 6 ft. side is 3 ft., what is the area of the second parallelogram?

23. The volumes of similar solids (solids of same shape, corresponding edges proportional) are as the cubes of corresponding edges, or of corresponding dimensions. Show by a sketch what this means.

24. If a watermelon 6 in. across costs 15 cents, what would

another cost at the same rate, whose dimension corresponding to the 6 in. line is 10 inches?

25. A box one of whose edges is 18 in. holds 3 bu. What would a box of the same shape hold, if the corresponding edge were 63 inches long? $2\frac{1}{3}$ inches long?

CHAPTER XXI

LINEAR EQUATIONS CONTAINING TWO UNKNOWN NUMBERS

§ 75. Plotting Linear Equations

1. The sum of two numbers is 4, their difference is 2. Find the numbers.

2. The sum of two numbers is 5. Find the numbers.

SOLUTION: The equation is $x+y=5$,

$$y=5-x.$$

Let x be some number, as 1, then $y=4$;

if $x=2$; if $x=5$; if $x=6\frac{1}{2}$;

then $y=3$; then $y=0$; then $y=-1\frac{1}{2}$, etc.

x	y
1	4
2	3
5	0
$6\frac{1}{2}$	$-1\frac{1}{2}$

Assume five other values of x and compute the corresponding values of y. Tabulate as above.

It is seen that, if a linear equation contains two unknown numbers, for every value of one of the unknown numbers there is a value of the other. Every pair of numbers in the tables satisfies the given equation. Hence, there is an indefinitely large number of solutions of a *single equation containing two unknown numbers.*

3. Find the pair of numbers which satisfies both equations of:

(1) $x+y=5$,
$2x+3y=12$;

(2) $5x+2y=1$;
$8x-y=10$;

(3) $3x-2x=-4$,
$2x-y=-1$;

(4) $3x+2y=26$,
$5x+9y=83$.

§ 76. Graphical Solution of Simultaneous Equations of the First Degree

1. Given (1) $x+2y=11$,

and (2) $2x+y=13$.

Assign to x any three or more values and from each equation find the corresponding value of y, thus getting several solutions. Tabulate the results. On squared paper find the *graphs* of the equations of (1) and (2).

Do the co-ordinates of points on the graph of equation (1) satisfy equation (1)? Equation (2)?

NOTE.—*The x- and y-distances are the co-ordinates.*

Do the co-ordinates of points on the graph of equation (2) satisfy equation (2)? Equation (1)? From the graphs find the solution which satisfies both equations (1) and (2).

Solve in a similar way by graph the following pairs:

2. $3x-4y=7$,
   $x+2y=9$.

3. $2x+3y=19$,
   $3x+2y=16$.

4. $3x-\frac{2}{3}y=3$,
   $\frac{2x}{5}-2y=4$.

§ 77. Equivalent Equations

1. Draw graphs of $x-y=2$,

and $2x=2y+4$.

Is there a common solution of these equations? Two *linear equations* with two unknowns having all of their solutions in common are called *equivalent equations.* How can one of these equations be derived from the other?

2. Are the equations $2x-y=3$?

and $6x=9+3y$ equivalent?

Answer the latter question both *graphically* and *algebraically.*

3. Make up two equivalent equations. Prove them equivalent.

§ 78. Inconsistent Equations

1. Draw graphs of $\begin{cases} 2x + y = 6, \\ 4x + 2y = 4. \end{cases}$

Is there a common solution for these equations?

Two linear equations with two unknowns having no common solution are called *inconsistent equations.*

2. Show by graph that $5x - y = 10$

and $5x = 2 + y$ are inconsistent.

§ 79. Elimination

1. Solve $5x + 4y = 22$

and $3x + y = 9$, for x and y.

To eliminate x (i. e., to remove x), multiply the first equation through by 3, and the second by 5. The resulting equations:

$$15x + 12y = 66,$$
$$15x + 5y = 45,$$

are equivalent, respectively, to the two given equations. Therefore, they have the same solution as the two given equations. Subtracting member from member, there results

$$7y = 21,$$
$$\text{or } y = 3.$$

The value of x may be found from the given equations by multiplying the second of the given equations through by 4, and subtracting as before. Do this.

Or, we may find the value of x by replacing, in the second of the given equations, y by its value, 3, which gives:

$$3x + 3 = 9,$$

whence $$3x = 6,$$

and $$x = 2.$$

2. Solve *by elimination* the following simultaneous equations:

(1) $\begin{cases} 3x+2y=12, \\ 4x-3y=-1; \end{cases}$

(2) $\begin{cases} 5a-2b=1, \\ 8a=5b-11; \end{cases}$

(3) $\begin{cases} \dfrac{2a}{3}-\dfrac{b}{2}=5, \\ \dfrac{a}{2}+\dfrac{2b}{3}=20; \end{cases}$

(4) $\begin{cases} 11t-10v=14, \\ 5t+\ 7v=41; \end{cases}$

(5) $\begin{cases} \dfrac{-5v}{8}+7t=13, \\ \dfrac{11v}{12}-\dfrac{5t}{8}=12; \end{cases}$

(6) $\begin{cases} 1.5x-3.7y=5.4, \\ x+3.5y=6; \end{cases}$

(7) $\begin{cases} ax+by=1, \\ bx+ay=1; \end{cases}$

(8) $\begin{cases} \dfrac{1}{x}+\dfrac{2}{y}=2, \\ \dfrac{2}{x}-\dfrac{2}{y}=1; \end{cases}$

(9) $\begin{cases} \dfrac{1}{1+x}+\dfrac{1}{1+y}=1, \\ \dfrac{2}{1+x}-\dfrac{1}{1+y}=\frac{1}{2}; \end{cases}$

(10) $\begin{cases} \dfrac{5x-2}{3x-1}=\dfrac{5y+7}{3y+16}, \\ \dfrac{x-1}{3x+5}=\dfrac{2y-5}{6y+3}. \end{cases}$

3. The rainfall in a certain locality one year was $\frac{17}{20}$ as much as it was the year after; and the total rainfall for the two years was 18.5 inches. Determine the amount for each year?

4. A man who can row 6 miles an hour down stream can row two miles an hour up stream. What is the speed of the current? What is the speed of rowing in still water?

5. By means of a chain running over the sprocket wheels of a bicycle the small wheel is driven by the larger one. The small wheel makes 90 revolutions a minute more than the large one. If the small wheel were $\frac{1}{4}$ smaller than it is, it would make 120 revolutions more than the large one. How many revolutions per minute does each sprocket wheel make?

6. A man and a boy together receive a wage of \$37.50. The man works 10 days and the boy 8 days. The man earns in 4 days \$3.50 more than the boy earns in 6 days. What amount does each receive?

7. If the altitude of a rectangle be increased by 4 inches and its base be diminished by 2 inches the area will be increased by 22 square inches. If the altitude be increased by 1 inch and the base be diminished by 1 inch, the area will be increased 2 square inches. Find the base and altitude of the given rectangle.

8. A bar of iron of uniform thickness 10 ft. long and weighing $1\frac{1}{2}$ cwt. is supported at its extremities in a horizontal position, and carries a weight of 4 cwt. suspended from a point 3 feet from one extremity. Find the pressure on the points of support. (Take the *turning-point* at the middle point of the beam.)

9. A bar, weighing 7 lb. per linear foot rests on a fulcrum three feet from one end. What must be its length, that a weight of $71\frac{1}{2}$ lb. suspended from that end may just be balanced by 20 lb. suspended from the other? What is the pressure at the fulcrum? (Take the turning-point at the midpoint.)

10. A beam of uniform thickness weighing 10 lb. is supported at the ends by the two props. A weight of 31 lb. hangs 4 ft. from one end. Find the length of the beam and where a weight must be placed that the pressure on the two props may be 18 and 25 pounds.

11. A signboard four feet long hangs in a horizontal position by hooks at its ends to an iron rod weighing 14 lb. The rod is supported at its ends. If the signboard weighs 32 lb. how far from the ends of the rod must the hooks be placed that the pressure at the ends of the rod may be 12 lb. and 34 lb.?

12. Write out the equations, from the laws of force, and find, for each of the six problems in the table below, the force, or arm not given in the table.

	d_1	F_1	d_2	F_2	d_3	F_3
I	8		−5	8	0	
II		6	−10	2	2	−5
III	4	2	3	9		−7
IV	−2	11		8	−3	6
V	8	−2	2		2	3
VI	7	5	2	−5	−5	

CHAPTER XXII

QUADRATIC EQUATIONS

§ 80. Uses of Quadratics and Interpretation of Negative Roots

1. A body is thrown vertically upward with a speed of $u(=50)$ feet per second. It passes a point $d(=25)$ feet from ground twice, first, going upward, and second, on its return. Its speed v at this point can be found from the equation

$$v^2=u^2-2gd=50\times 50-64\cdot 25\,,$$

Hence $$v^2=900\,.$$

Extracting the square roots of both sides of this equation gives $v=30$ and $v=-30$. The first being the speed of the particle when moving in the direction in which it was thrown, the second its speed when moving in the opposite direction on its return.

2. Find the value of x in $x^2=4$.

Extracting the square root of both sides of the equation, gives $$x=2,\text{ or } x=-2.$$

Check by substituting each one of these values in the original equation.

3. Solve for x:

(1) $x^2=625$;

(2) $x^2-121=0$;

(3) $3x^2-2700=0$;

(4) $\dfrac{x^2}{5}=5$;

(5) $\dfrac{2x^2}{7}=14$.

4. If a particle is dropped so that it can fall freely, the attraction of the earth increases its speed by 32 feet per second. This increase is denoted by the letter g. If in t seconds the particle has moved s feet, then $s=\frac{1}{2}gt^2$. Its speed v is found from the equation $v^2=2gs$.

(1) How long will it take a ball to fall 2,304 feet?

(2) How long will it take a body to fall 1,600 feet?

(4) What would be the velocity with which the body in the last problem would strike the earth?

(4) A body falls from rest. How long must it move to acquire a velocity of 836 feet a second?

5. If a force of P pounds acting on a body of weight W moved it through a distance of s feet, the velocity acquired can be found from the equation: $Ps=\frac{Wv^2}{2g}$.

A hammer, weighing 16 lb., strikes a post with a force of 480 lb., and drives it $\frac{1}{10}$ of an inch. Find the velocity with which the hammer moved at the instant it strikes the post.

6. The law of vibration of a pendulum is $gt^2=\pi^2 l$. In what time will a pendulum vibrate whose length is 15 inches?

§ 81. Formal Problems

Solve the following equations:

1. $(x+4)^2=9$; 2. $(x+k)^2=9$.

In exercises 3–11, add something to both sides that will make the first member of the following equations a perfect trinomial square and find the values of the unknown:

3. $x^2+2x=8$;
4. $x^2+6x=-5$;
5. $x^2+2ax=16-a^2$;
6. $x^2+4x-1=0$;
7. $y^2-3y+1=0$;
8. $2n^2-3n-2=0$;
9. $5r^2+11=3r$;
10. $6y^2+5ay=-a$;

11. Solve $ax^2+bx+c=0$. Result, $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Definition.—A more scientific meaning of the phrase to "solve an equation" is to find the root, or the roots, of the equation. A root of an equation is the value, or values, of the unknowns in terms of the coefficients.

12. Solve the following by use of the formula in Problem 11:

(1) $x^2-5x+6=0$;
(2) $2x^2+9=9x$;
(3) $x^2+22(x+5)=0$;
(4) $21+x=2x^2$;
(5) $mx^2+nx+p=0$;
(6) $3u^2-4u-10=0$;
(7) $24x-x^2=0$;
(8) $6x^2-5x=6$;
(9) $7t^2-37t=-10$;
(10) $2s^2-13s=-20$.

13. Solve by factoring:

(1) $x^2-2x=15$;
(2) $x^2-5x+6=0$;
(3) $6x^2-x-1=0$;
(4) $2x^2-7x-15=0$;
(5) $x(x+7)=7(x+28)$;
(6) $(x-1)(x+1)(x-2)=0$;
(7) $x^3+x^2-x-1=0$;
(8) $x^2+ax+bx+ab=0$;
(9) $6y^2-32y=-32$;
(10) $6p^2+5p=6$.

§ 82. Problems Leading to Quadratics

In the following obtain two different expressions for some number, or magnitude; write the expressions equal to each other, and solve the resulting equation. Finally, test the correctness of your results by substituting in the equation first formed.

1. One side of a rectangle is 3 ft. longer than the other. If the longer side be diminished by one foot and the other side increased by one foot the area of the rectangle will be 30 sq. ft. How long is the rectangle?

2. The area of a rectangle is 54 sq. in. and the sum of its length and breadth is 15 inches. How long is the rectangle?

3. Find the length of a rectangle whose area is 60 sq. in. and the sum of whose breadth and length is 16 inches.

4. The diagonal and the longer side of a rectangle are together five times the shorter side, and the longer side exceeds the shorter by 33 yards. What is the area of the rectangle?

PRINCIPLE.—*In a right triangle the square on the hypothenuse equals the sum of the squares on the other two sides.*

5. The hypothenuse of a right triangle is 5 in. and one of the sides is 1 inch longer than the other. Find the length of the sides.

6. Two trains 100 miles apart on roads which cross at right angles are running toward the crossing. One train runs 10 mi. an hour faster than the other. At what rates must they run, if they both reach the crossing in two hours?

7. A tree was broken by a storm so that the top touched the ground 50 ft. from the foot of the tree. The stump was $\frac{3}{8}$ of the height of the tree. What was the height of the tree?

8. Two trains move from a crossing in perpendicular directions. One has a speed of 12 mi. per hr., the other of 16 mi. per hour. In how many hours will they be 70 mi. apart?

9. If v is the velocity, t the time, s the distance passed over by a body thrown vertically downward, then (1) $v=u+gt$ and (2) $s=ut+\frac{1}{2}gt^2$, where u is the velocity the body has at the start (*initial* velocity). If the body is thrown vertically upward, the following laws apply: (1) $v=u-gt$; (2) $s=ut-\frac{1}{2}gt^2$; and (3) $s=\dfrac{u^2-v^2}{2g}$.

(1) With what initial velocity must a bullet be fired upward that it may rise to a height of 6,400 feet?

(2) A body is projected upward with a velocity of 80 ft. per second; in what time will it return to hand? Answer: In 5 seconds.

(3) A falling body starts with a velocity of 30 ft. a second. When will it have gone just 300 feet?

(4) A ball is thrown vertically into the air with a velocity of 40 ft. a second. When will it be at the height of 16 feet?

(5) How hard must I throw a ball that it shall just reach a man on a scaffold 25 ft. above me?

CHAPTER XXIII

LOGARITHMS

§ 83. Logarithms of Exact Powers of 10

NOTE.—The teacher is to work this chapter through with *the class*. Review here the ideas "factor," "power," "root," and "exponent."

Express 100 *as factors* in 10; thus, $100 = 10 \times 10$. Express 1,000 similarly; 10,000; 100,000.

Express 100, 1,000, 10,000, and 100,000 as *powers* of 10; thus $100 = 10^2$, etc.

What has been done may be shown thus:

The Number	Expressed as Factors	Expressed as Powers of 10
100	10×10	10^2
1,000	$10 \times 10 \times 10$	10^3
10,000	$10 \times 10 \times 10 \times 10$	10^4
100,000	$10 \times 10 \times 10 \times 10 \times 10$	10^5

What does the small figure, placed at the right and above the 10's in the last column express?

This number written in small figures to the right of and above the 10, is called the *exponent of the power*.

Distinguish between the *power* and the *exponent* in each of the following numbers: 2^2, 2^3, 2^4, 3^2, 3^3, 4^2, 4^3, 5^3, a^2, a^3, x^4.

DEFINITION.—The number (exponent) which expresses how many times 10 must be used as a factor to give a certain number is called the *logarithm of the latter number*. Thus, in $1{,}000 = 10^3$, the 3 is the logarithm of 1,000. Give other examples from the foregoing table. Give the logarithm of 1,000,000; of 1 with 7 zeros following; of 1 with 12 zeros following, etc.

Fill out the logarithm column of the following table:

The Number	Power	Logarithm
100	10^2	2
1,000	10^3	
10,000	10^4	
100,000	10^5	
1,000,000	10^6	

Starting with the fourth number of column 1, how may the third number of the same column be obtained from it? The first from the second number? What number, following the same law, could be added above 100 in column 1? Write it in its proper place.

What second number in column 1 could be added above the 10, by following the same law? Write it in column 1 also.

State in words the law of the numbers of column 1.

LAW STATED.—*Any number of column 1 may be obtained from the number next below it by dividing the number next below by 10.*

State the same law using the word "multiplying" instead of "dividing," making such other changes in the wording as are necessary.

Starting with the last number of column 3, how is the next to the last obtained from it? The second from the third number? The first from the second?

Add another number above the 2, following the law of the numbers of column 3. Add a second number above this "1."

State the law by which each number of column 3 is obtained from the number next below it; from the number next above it.

What then is the logarithm of 10? Of 1?

Arrange what has been found thus:

Number	Power	Logarithm
1	10^0	0
10	10^1	1
100	10^2	2
1,000	10^3	3
10,000	10^4	4
100,000	10^5	5

If a number lies between 10 and 100, between what two numbers must its logarithm lie? Between 1 and 10? 100 and 1,000? 1,000 and 10,000?

What kind of numbers lie between 0 and 1? Answer: Fractions less than 1. Between 1 and 2? Between 2 and 3? 3 and 4? 4 and 5? Give examples of numbers lying in each of the intervals just mentioned.

Give examples of numbers that lie between 1 and 10? 10 and 100? 100 and 1,000? 1,000 and 10,000? How may numbers that lie in each of the last-mentioned intervals be recognized?

The logarithm of a certain number is 1.684. Between what two numbers of column 1 of the table does this number lie?

State between what two numbers of column 1 numbers having the following logarithms lie: (1) 2.672; (2) .486; (3) 3.125; (4) 4.061; (5) .186; (6) .006.

Between what two numbers of column 3 must the logarithms of the following numbers lie: (1) 18; (2) 7.84; (3) 25.74; (4) 3.62; (5) 268.7; (6) 1,286; (7) 68,491?

The question now arises how may the logarithms of numbers that are not exact powers of 10 be obtained?

§ 84. Logarithms of Numbers that are not Exact Powers of 10

How may the next to the last number (the 4) of column 3 of the last table above be found from the last (the 5)?

How may the next preceding (i. e., the 3) be found from the next to the last (the 4)?

How may each number of column 3 be found from the number last below it? From the number next above it?

What, then, is the law of the numbers of column 3?

Answer: *Each number of column 3 may be obtained by adding a constant number (i. e., 1) to the number next above it.*

State the same law using the word "subtracting" for "adding," making other necessary changes in the wording.

How then might another number be put in between each pair of numbers of column 3 following a similar law?

Answer: By adding .5 to each number of column 3 excepting the last. We should obtain thus: 0, .5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5.

In a similar way, we could obtain, by putting other numbers between these: 0, .25, .50, .75, 1, 1.25, 1.50, 1.75, 2, 2.25, 2.50, 2.75, 3, 3.25, 3.50, 3.75, 4, 4.25, 4.50, 4.75, 5.

Let us now see whether we may insert other numbers between those of column 1 in accordance with the law of the numbers already in column 1, with the aid of a *factor different from 10.*

To this end let us write down the law of the numbers already in column 1. How may each number of the column be obtained from the number next below it? The next above it?

What is the law of the numbers of column 1?

Answer: *Each may be obtained by multiplying the one next above it by a certain number (i. e., 10), or by dividing the number next below it by this same number (10).*

For the new series of numbers we wish to obtain by putting other numbers between those of column 1, let the constant multiplier (or constant divisor) be x. For the number between 10 and 100 we should have $10x$, and multiplying this $10x$ again by x, the product must then be 100. (why?)

Or we have:

$$10x^2 = 100, \text{ whence } x^2 = 10. \quad \text{(Why?)}$$

This requires the solution of a quadratic equation.

Extracting the square root of 10, we find $x = 3.1623$ (omitting the negative result, -3.1623).

We now have the new series: 1, 3.16, 10, 31.62, 100, 316.23, 1,000, 3,162.3, 10,000.

How are 31.62, 316.23, and 3,162.3 found?

We may again fit numbers between these by taking our multiplier y, and noticing that between 10 and 31.62, for example, we should have $10y$ and also, that $10y^2=31.62$, or that $y^2=3.1623$ (why?).

Whence, extracting the square root of 3.1623, we find $y=1.7785$ (again omitting the -1.7785).

This gives the series: 1, 1.78, 3.16, 5.62, 10, 17.78, 31.62, 56.23, 100, 177.85, 316.23, 562.34, 1,000, 1778.50, 3,162.30, 5,623.40, 10,000.

How are 5.62, 17.78, 56.23, 177.85, 316.23, etc., found?

Arranging these numbers in a table, with a third column to show their meanings, we have:

Numbers	Logarithms	Meanings
1.00	0.00	$1.00=10^{.00}$
1.78	0.25	$1.78=10^{.25}$
3.16	0.50	$3.16=10^{.50}$
5.62	0.75	$5.62=10^{.75}$
10.00	1.00	$10.00=10^{1.00}$
17.78	1.25	$17.78=10^{1.25}$
31.62	1.50	$31.62=10^{1.50}$
56.23	1.75	$56.23=10^{1.75}$
100.00	2.00	$100.00=10^{2.00}$
177.85	2.25	$177.85=10^{2.25}$
316.23	2.50	$316.23=10^{2.50}$
562.34	2.75	$562.34=10^{2.75}$
1,000.00	3.00	$1,000.00=10^{3.00}$
1,778.50	3.25	$1,778.50=10^{3.25}$
3,162.30	3.50	$3,162.30=10^{3.50}$
5,623.40	3.75	$5,623.40=10^{3.75}$
10,000.00	4.00	$10,000.00=10^{4.00}$

Evidently other numbers might be inserted in columns 1 and 2 and the table extended indefinitely. Such extensions

have been made by computers, and the numbers together with their logarithms have been arranged in tables convenient for use. It is obvious that such a table could be extended either downward, or by the insertion of other numbers between those given in the table. Carrying the extension one step farther, we have the following table of numbers and their logarithms.

Number	Logarithm	Number	Logarithm
1.00000.........	.00000	5.62340.........	.75000
1.07461.........	.03125	6.04296.........	.78125
1.15478.........	.06250	6.49382.........	.81250
1.24094.........	.09375	6.97830.........	.84375
1.33352.........	.12500	7.49894.........	.87500
1.43302.........	.15625	8.05842.........	.90625
1.53993.........	.18750	8.65964.........	.93750
1.65482.........	.21875	9.30572.........	.96875
1.77828.........	.25000	10.00000.........	1.00000
1.91205.........	.28125	10.74610.........	1.03125
2.05352.........	.31250	11.54780.........	1.06250
2.20673.........	.34375	12.40940.........	1.09375
2.37137.........	.37500	13.33520.........	1.12500
2.54830.........	.40625	14.33020.........	1.15625
2.73842.........	.43750	15.39930.......	1.18750
2.94263.........	.46875	16.54820.........	1.21875
3.16229.........	.50000	17.78280.......	1.25000
3.39821.........	.53125	19.12050.........	1.28125
3.65174.........	.56250	20.53520.........	1.31250
3.92419.........	.59375	22.06730.........	1.34375
4.21695.........	.62500	23.71370...	1.37500
4.53158.........	.65625	25.48300.........	1.40625
4.86968.........	.68750	27.38420.........	1.43750
5.23299.........	.71875	29.42630.........	1.46875

The tables to be used, called *Four-Place Tables* by Beebe, contain the fractional parts (called mantissas) of the logarithms

to four places of decimals of the numbers 1 to 10,000. Members of the class will provide themselves with a copy of these tables, or other four-place tables.

§ 85. Meaning and Use of a Four-Place Table of Logarithms

A four-place table of common logarithms contains the fractional parts (the mantissas) of the logarithms of all numbers from 10 to 1,000. The integral part (called *the characteristic*) of the logarithm is easily supplied by recalling that all numbers between 1 and 10 have one digit on the left of the decimal point, and that their logarithms are all 0+ a *fraction less than 1*, etc.

Give examples from a table of four-place logarithms.

§ 86. To Find from a Table the Logarithm of a Given Number

(1) Let it be required to find from a table the logarithms of 28.6 and of 326.8.

The mantissa of the logarithm of 28.6 is found in the horizontal line in which 28 stands and in the vertical column at the top of which 6 stands. The mantissa is .4564.

Since 28.6 lies between 1 and 100, the characteristic is 1. The entire logarithm is then 1.4564. The meaning of this is shown by the *explanatory equation:* $10^{1.4564}=28.6$.

(2) To find the logarthim of 326.8.

First find the mantissa of the logarithm of 326 as before. It is .5132.

Since the number 326.8 lies between 326 and 327 the mantissa must lie between the mantissas .5132 and .5145 (the mantissa of 327). The interval between 5132 and 5145 is 13. It is assumed that the correct mantissa of the logarithm of 326.8 lies between 5132 and 5145 just as 326.8 lies between 326 and 327. But 326.8 lies .8 of the way from 326 up toward 327. But .8 of 13 equals 10.4, or 10, *nearly*. Adding 10 to

5132, we have 5142. The mantissa of the logarithm of 326.8 is, then, .5142.

Since 326.8 lies between 100 ($=10^2$) and 1,000 ($=10^3$), the characteristic must be 2, and the entire logarithm of 326.8 is 2.5142. The *explanatory equation* is: $10^{2.5142}=326.8$.

Observe that to find the characteristic of the logarithm of a number it is necessary merely to notice between which two perfect powers of 10 the given number lies.

(3) Find logarithms of the following numbers: (1) 13.6; (2) 60.5; (3) 75.22; (4) 186.2; (5) 765.3; (6) 3.146; (7) 1.862; (8) 2.372; (9) 12.09.

§ 87. To Find from a Table the Number that Corresponds to a Given Logarithm

1. Let it be required to find the number that corresponds to the logarithms, 1.6149 and 2.5394.

Glancing into the body of the table the mantissa, .6149, is found in the horizontal line in which 41 stands at the left end and 2 stands at the top of the vertical column. The *sequence of digits* (succession of figures) that compose the required number is, then, 412.

Since the characteristic of the given logarithm (1.6149) is 1, the sequence of digits must *be so pointed* as to make the number fall between 10 and 100. The number must then be 41.2. *Explanatory equation* $10^{1.6149}=41.2$.

(2) To find the number corresponding to the logarithm, 2.5394.

Entering the body of the table, take out the nearest mantissa to 5394 that can be found, and write down the first three digits as was done above. The tabular mantissa nearest to 5394 is 5391. The first three digits corresponding to 5391, in order, are 346.

The mantissa next larger than 5394 is 5403.

Arrange the numbers thus:

$$\begin{array}{lll} \text{Mantissa of } & & 346 = .5391 \\ \text{"} & \text{"} & ? = .5394 \\ \text{"} & \text{"} & 347 = .5403 \end{array}$$

This scheme of arrangement shows the required *sequence of digits* to be between 346 and 347. Hence, the first three digits sought must be 346. Why? From 5391 to 5403 is 12, and from 5391 to 5394 is 3. We wish then to put a sequence of digits between 346 and 347, and $\frac{3}{12}$, or $\frac{1}{4}$, of the way from 346 to 347. Evidently, the sequence desired must be 34625.

This sequence must be *so pointed* as to give a number between 10^2 (or 100) and 10^3 (or 1,000), because the given logarithm is 2.5394 and must correspond to a number in the interval from 100 to 1,000.

The required number is, then, 346.25.

3. Find numbers corresponding to the following logarithms, and write the explanatory equations: (1) 1.2672; (2) 1.7566; (3) 2.4942; (4) 3.4871; (5) 3.4876; (6) 2.6520; (7) 1.9507; (8) 1.9076; (9) 2.9673; (10) 3.9536; (11) 3.9387; (12) 2.8630; (13) .8235; (14) .8509; (15) .8569.

§ 88. Questions and Problems

1. Show from a four-place table the complete logarithm (characteristic and mantissa) of 47 and write the explanatory equation.

2. Show the complete logarithms and write the explanatory equations for the following: 35; 49.5; 54; 59; 68.5; 73; 86.5; 97.5; 9.

3. To find the logarithm of the product of 67×48.

SOLUTION: From the table, log. $7 = .8491$, log. $8 = .9031$, and log. $56 = \text{log.}\ (7 \times 8) = .8451 + .9031 = 1.7482$.

Also from the table, log. $48 = 1.8812$, log. $67 = 1.8261$.

Then $$48 = 10^{1.6812}$$
$$67 = 10^{1.8261}$$
and $$48 \times 67 (= 3216) = 10^{1.6812} \times 10^{1.8261} = 10^{1.6812 + 1.8261}$$
$$= 10^{3.5073}; \text{ just as } 10^2 \times 10^3 = 10^5.$$

log. $(48 \times 67) =$ log. $3216 =$ log. $48 +$ log. $67 = 1.8612 + 1.8261$ $= 3.5073$.

These examples illustrate that *the logarithm of a product equals the sum of the logarithms of the factors.*

The student may give further examples from the table.

4. Find the logarithms of the following products: (1) 37×59; (2) 69×52.5; (3) 84×88.5; (4) 17×78.5; (5) $45 \times 28 \times 15$; (6) $16 \times 39 \times 48.5$.

5. To find the logarithm of a quotient from the logarithms of the numbers to be divided.

(1) Find the logarithm of $\frac{63}{9}$, or 7, from log. 63 and log. 9 by use of the table. From the table, log. $63 = 1.7993$, log. $9 = .9542$, and log. $7 (= \frac{63}{9}) = .6451$. But $1.7993 - .9542 = .6451$, or log. 63 − log. 9 = log. 7.

Again, from the table

$$63 = 10^{1.7993}$$
$$9 = 10^{.9542}$$

$$63 \div 9 = 10^{1.7993} \div 10^{.9542} = 10^{1.7993 - .9542} = 10^{.6451} = 7, \text{ just as } 10^5 - 10^2 = 10^3.$$

(2) Find the log. $\dfrac{68}{19.5}$, or $68 \div 19.5$.

From the table:

$$68 = 10^{1.8325}$$
$$19.5 = 10^{1.2900}$$

$$68 \div 19.5 = 10^{1.8325} \div 10^{1.2900} = 10^{1.8325 - 1.2900} = 10^{.5425} \text{ just as } 10^6 \div 10^4 = 10^{6-4} = 10^2.$$

Or, log. $(68 \div 19.5) =$ log. 68 − log. 19.5, which illustrates the principle that *the logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.*

The student may give further illustrations from the table.

6. Find the logarithms of the following quotients:

(1) $53 \div 27$; (2) $48 \div 13.5$; (3) $87.5 \div 24.5$; (4) $98.5 \div 14.5$; (5) $71.5 \div 49$.

7. Find the logarithm of the square of 37.5.

Solution.—Since $7 \times 7 = 49$ and log. $7 = .8451$ and log. $49 = 1.6902 = 2 \times .8451$, we have log. $7^2 = 2$ log. 7. Show with tables that log. $81 = 2$ log. 9.

Finally, $37.5 \times 37.5 = 10^{1.574} \times 10^{1.574} = 10^{2 \times 1.574} = 10^{3.148}$. The desired logarithm is 3.148, which is twice the logarithm of 37.5. This illustrates that *the logarithm of any power equals the logarithm of the base times the exponent of the power.*

8. Find the logarithm of the cube 37.5.

Solution: $37.5 \times 37.5 \times 37.5 = 10^{3 \times 1.574} = 10^{4.722}$, just as $10^2 \times 10^2 \times 10^2 = 10^{3 \times 2} = 10^6$.

9. Find the logarithms of the square, the cube, and the fourth power of the following: (1) 16.5; (2) 36.5; (3) 98; (4) 71; (5) 68.5; and (6) 99.5.

10. Find the logarithm of the square root of 33.

Solution: Let x denote the desired logarithm of the square root of 33. Then $10^x \times 10^x = 33$. But $33 = 10^{1.518}$, and $10^x \times 10^x = 10^{2x}$. Then $10^{2x} = 10^{1.518}$. This will be true if we make $2x = 1.518$ or $x = .759$, which is the desired logarithm.

Several such logarithms of selected tabular numbers may be found by the student.

It will be seen that we may state the following principle:

The logarithm of the square root of any number is one-half of the logarithm of the number.

In a similar way develop the principles for finding the logarithms of the cube root and of the fourth roots of numbers.

Then generalize a method for other roots and test the generalized method by problems.

The logarithms of some numbers may be found from the logarithms of other numbers. For example:

$4 = 2 \times 2 = 10^{0.3010} \times 10^{0.3010} = 10^{0.3010+0.3010} = 10^{0.6020}$,

$\therefore \log . \ 4 = 0.6020$.

$6 = 2 \times 3 = 10^{0.3010} \times 10^{0.4771} = 10^{0.3010+0.4771} = 10^{0.7781}$,

$\therefore \log . \ 6 = 0.778\dot{1}$.

$9 = 3 \times 3 = 10^{0.4771} \times 10^{0.4771} = 10^{0.4771+0.4771} = 10^{0.9542}$,

$\therefore \log . \ 9 = 0.9542$.

$14 = 2 \times 7 = 10^{0.3010} \times 10^{0.8451} = 10^{0.3010+0.8451} = 10^{1.1461}$,

$\therefore \log . \ 14 = 1.1461$.

$10 = 2 \times 5 = 10^{0.3010} \times 10^{0.6990} = 10^{0.3010+0.6990} = 10^{1.0000}$,

$\therefore \log . \ 10 = 1.0000$ (as we know).

$126 = 9 \times 14 = 10^{0.9542} \times 10^{1.1461} = 10^{0.9542+1.1461} = 10^{2.1003}$,

$\therefore \log . \ 126 = 2.1003$.

11. Given log. 2 = 0.3010; log. 3 = 0.4771; log. 5 = 0.6990; log. 7 = 0.8541; log. 11 = 1.0424; log. 13 = 1.1139. Find the logarithms of the following numbers:

(1) 6; (2) 9; (3) 4; (4) 8; (5) 12; (6) 14; (7) 15; (8) 21; (9) 35; (10) 33; (11) 42; (12) 26; (13) 39; (14) 55; (15) 77; (16) 65; (17) 143; (18) 30; (19) 70; (20) 105.

12. Find the following quotients by the use of logarithms:

(1) $\frac{18}{3}$; (2) $\frac{36}{12}$; (3) $\frac{165}{15}$; (4) $\frac{143}{11}$.

13. Find the indicated roots of the following by logarithms:

(1) $\sqrt{16}$; (2) $\sqrt[3]{27}$; (3) $\sqrt{64}$; (4) $\sqrt[4]{144}$; (5) $\sqrt[3]{1728}$; (6) $\sqrt[5]{628}$; (7) $\sqrt[3]{180}$.

§ 89. Problems for Solution by the Aid of Logarithmic Tables

Multiplication

1. A lot is 24.8 by 120.6; how many square feet are there in it?

2. A field is 38.4 rd. $\times$ 62.8 rd.; how many square rods in it?

3. A street car averages 18 trips a day, carrying 36 paid passengers a trip at 5 cents a fare for 28 days a month. What amount of money is taken in on this car during the month.

Log. 2=0.3010; log. 3=0.4771; log. 7=0.8451.

4. How many cubic feet in a box 2′.5×3′.6×4′.2?

5. How many cubic yards in a room 4.6 yd.×5.5 yd.× 13.3 yd.?

6. How many cu. ft. in the room of problem 5?

7. How many cu. ft. in a wagon box 2′.9×3′.5×10′.2?

8. Water weighs 62.5 lb. per cu. ft and copper is 8.8 times as heavy. How heavy is a mass of copper 2′.5×3′.87 ×4′.3?

9. Ice weighs .92 as much as an equal bulk of water. How much does a mass of ice 6′×108′×40′ weigh?

10. What will it cost to pave a street 24 yd. wide a distance of 1 mile at $4.78 per sq. yd?.

Division

1. How many sq. yd. in the lot of problem 1?

2. How many acres in the field of problem 2?

3. Find the weight in ounces of a cu. in. of iron, a cu. ft. weighing 487.5 pounds.

4. How many bu. of small grain will the wagon box of problem 7 contain if 1 bu.=1.2 cubic foot?

5. How many bu. of ear corn will the wagon box of problem 7 contain if 1 bu.=2.5 cubic feet?

6. How many bu. of ear corn will a crib 12′×16′.2×40′.4 hold if 2.5 cu. ft.=1 bushel?

7. A cu. ft. of steel weighs 490 lb. Find the weight of a cubic inch. How many cubic feet will weigh 1 ton (2,000 lb.)?

Involution

1. How many cubic feet in a 5-ft. cube? In an 8-ft. cube? In a 16-ft. cube?

2. How many cubic feet are there in a 4.5-ft. cube? In a 7.6-ft. cube? In a 24.6-foot cube?

3. How many cubic inches are there in a 36″.5 cube? In a 58″.6 cube? In a 63″.7 cube?

Evolution

1. Find the length of the edge of a cube whose volume is 216 cu. ft.; 343 cu. ft.; 729 cubic feet.

2, Find the approximate length of an edge of a cube whose volume is 74.09 cu. ft.; 1092.73 cu. ft.; 4330.75 cubic feet.

3. Find by logarithms approximately the length of one side of the following squares:

(1) 104.04 sq. ft.; (2) 357.21 sq. ft.; (3) 384.16 sq. ft.; (4) 1004.89 sq. ft.; (5) 1497.69 square feet.